اطلاعات و دوربری کوانتومی

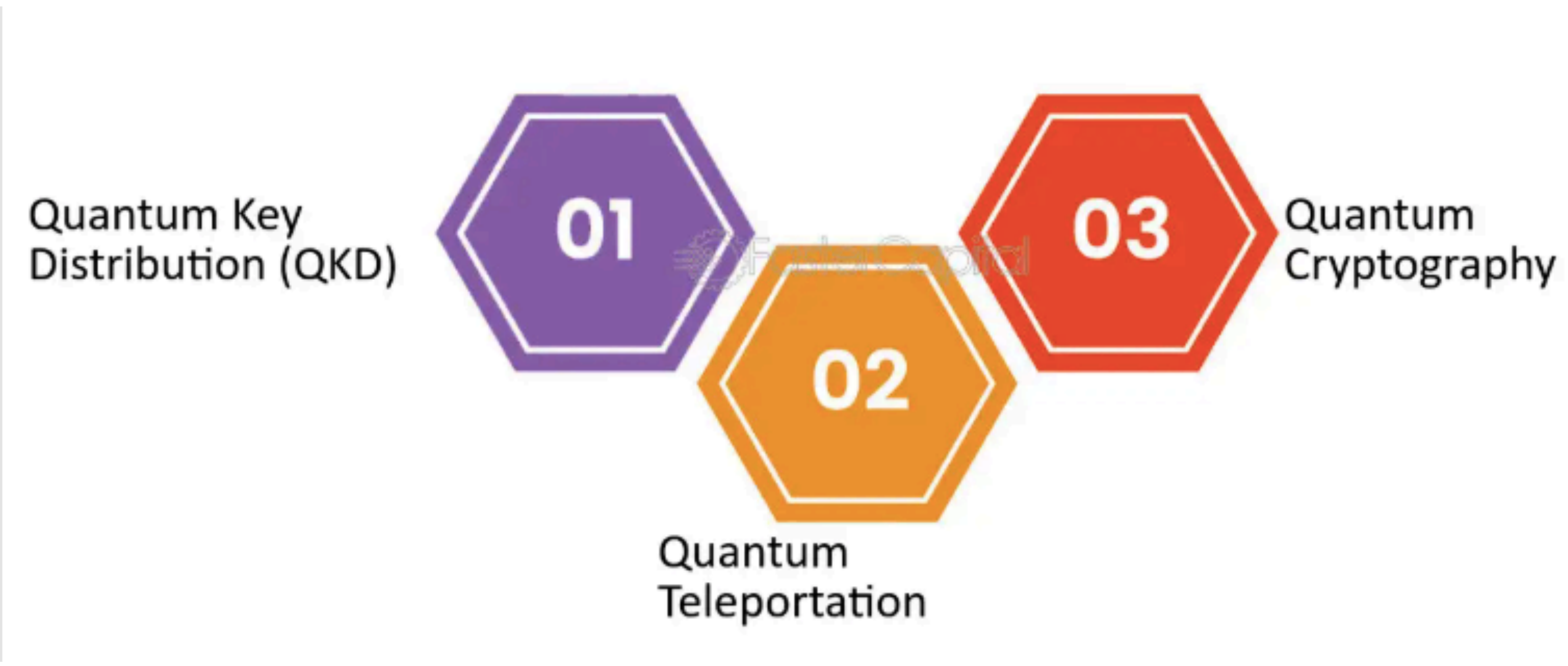
Quantum Information and Teleportation

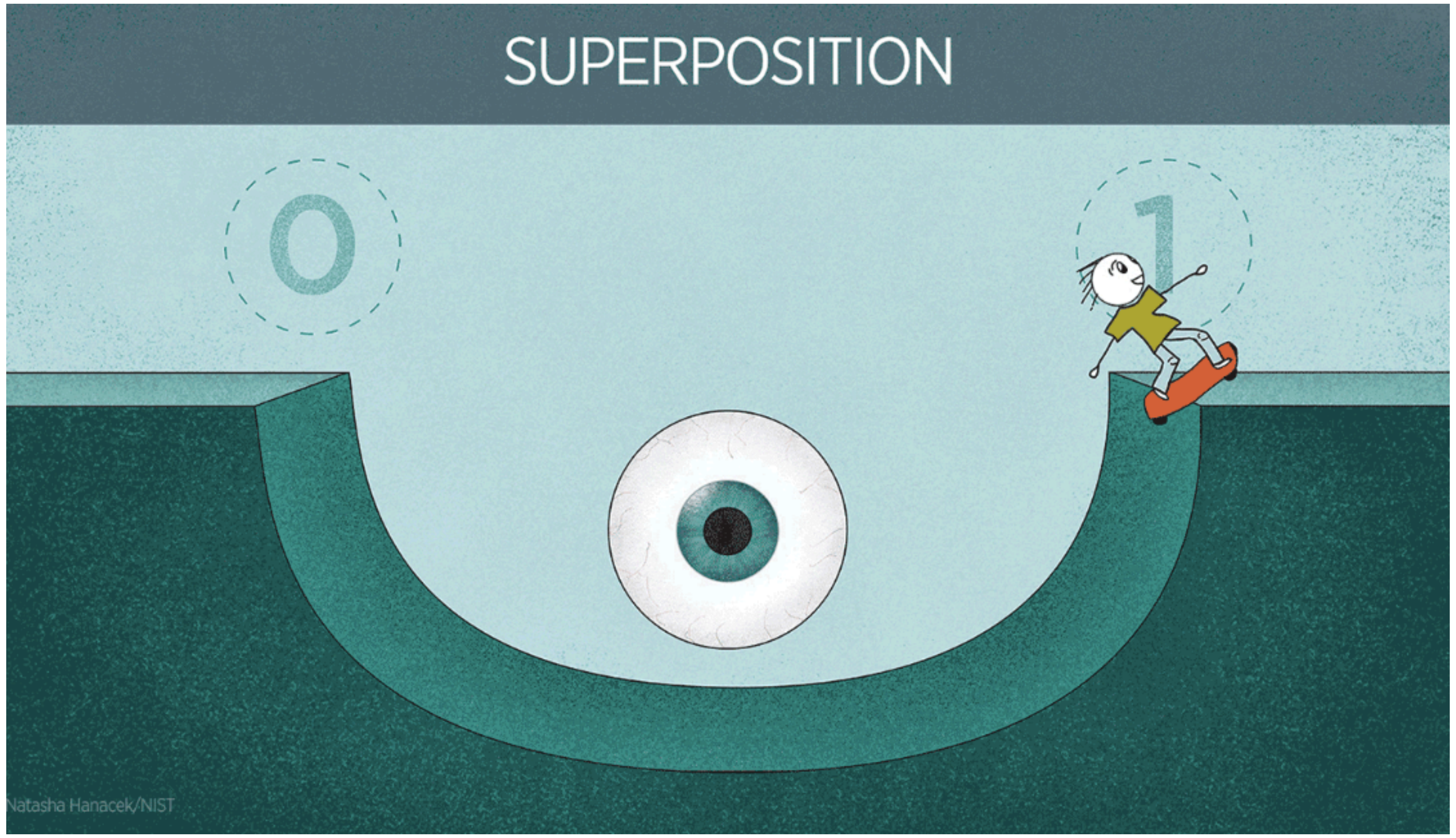
فرهاد فضیله

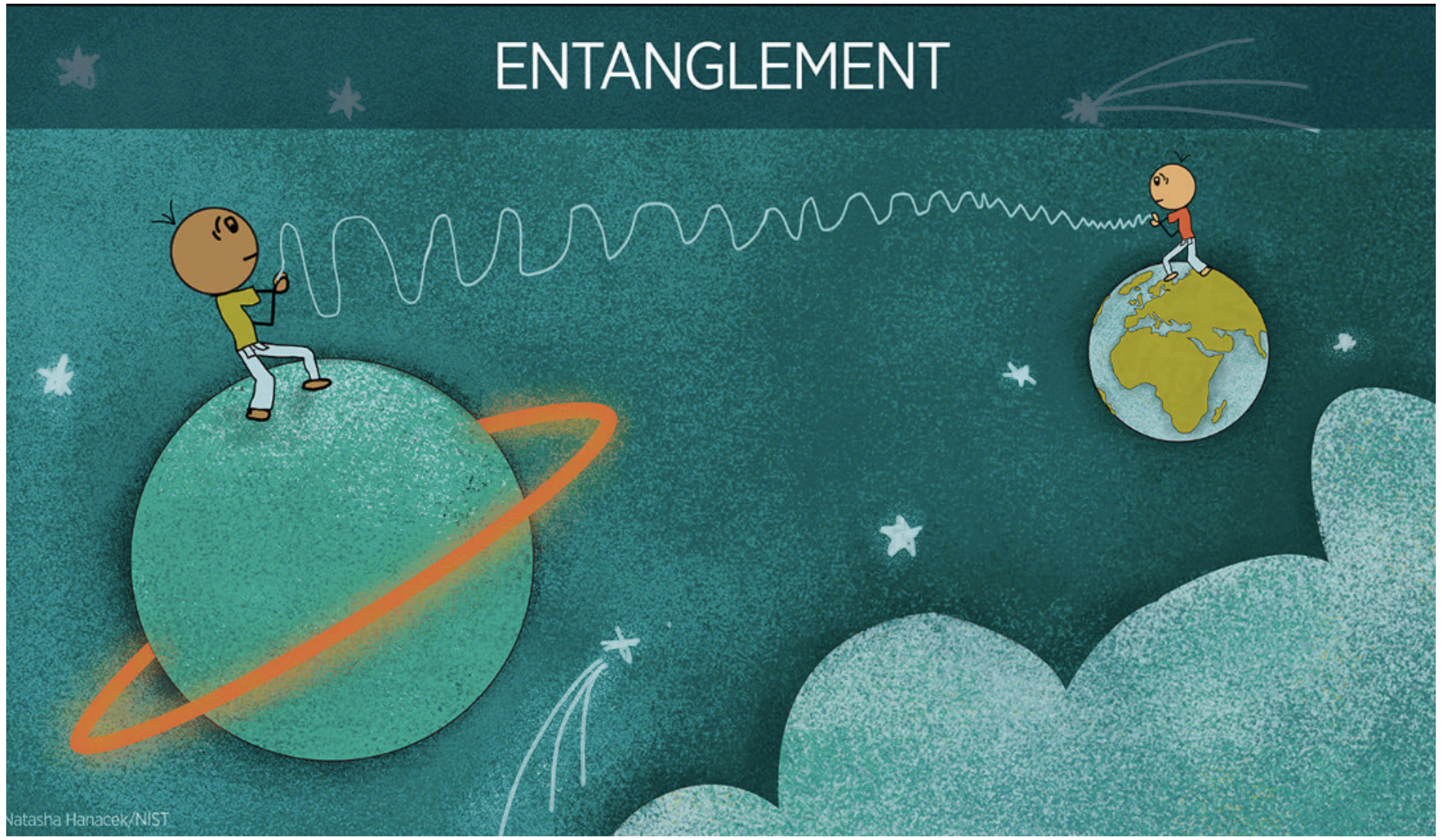
۱۴۰۴

Classical vs. Quantum

- Superposition
- Entanglement



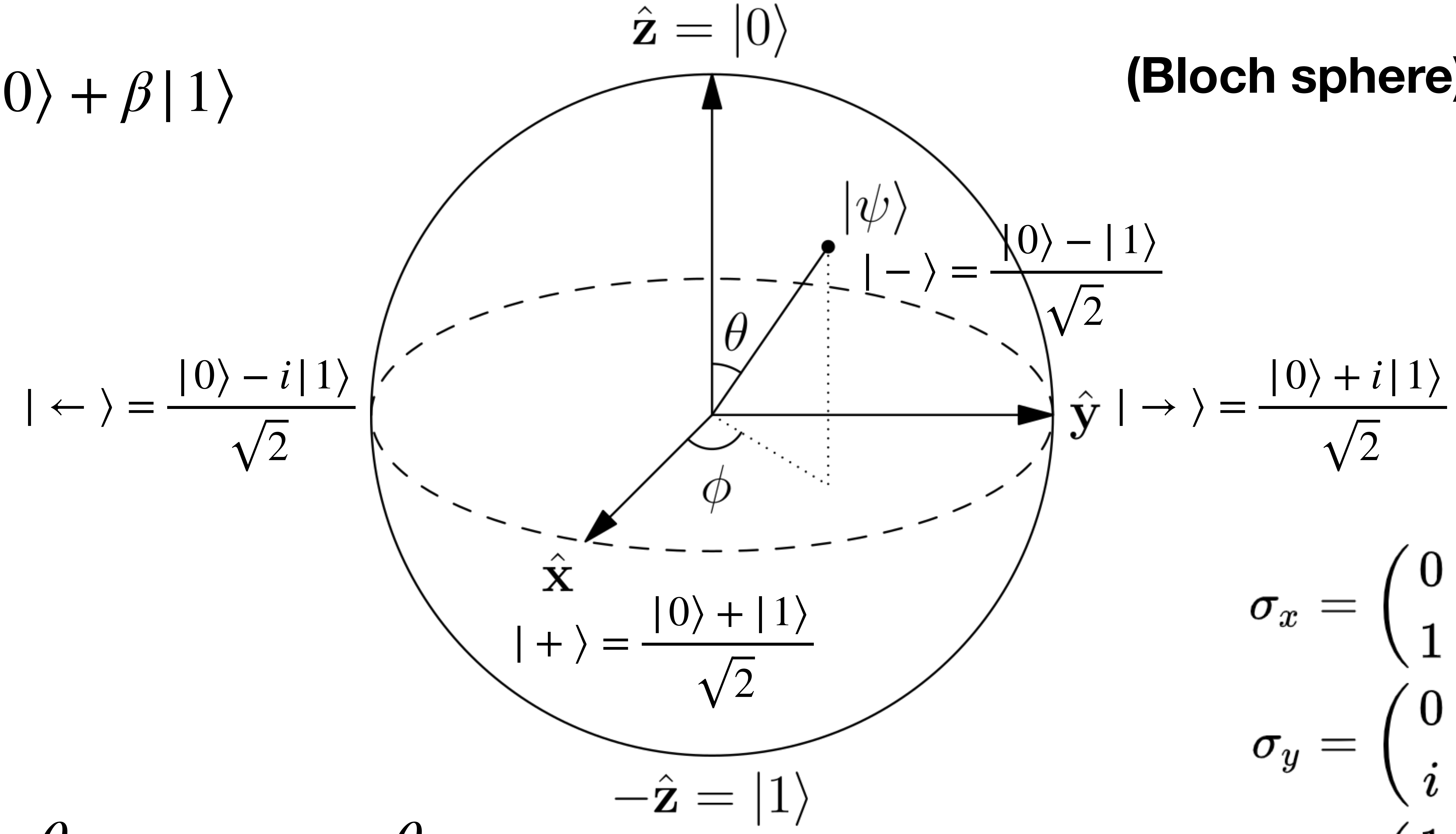




مروری بر مبانی نظری دروازه‌های کوانتومی

کره بلاخ (Bloch sphere)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

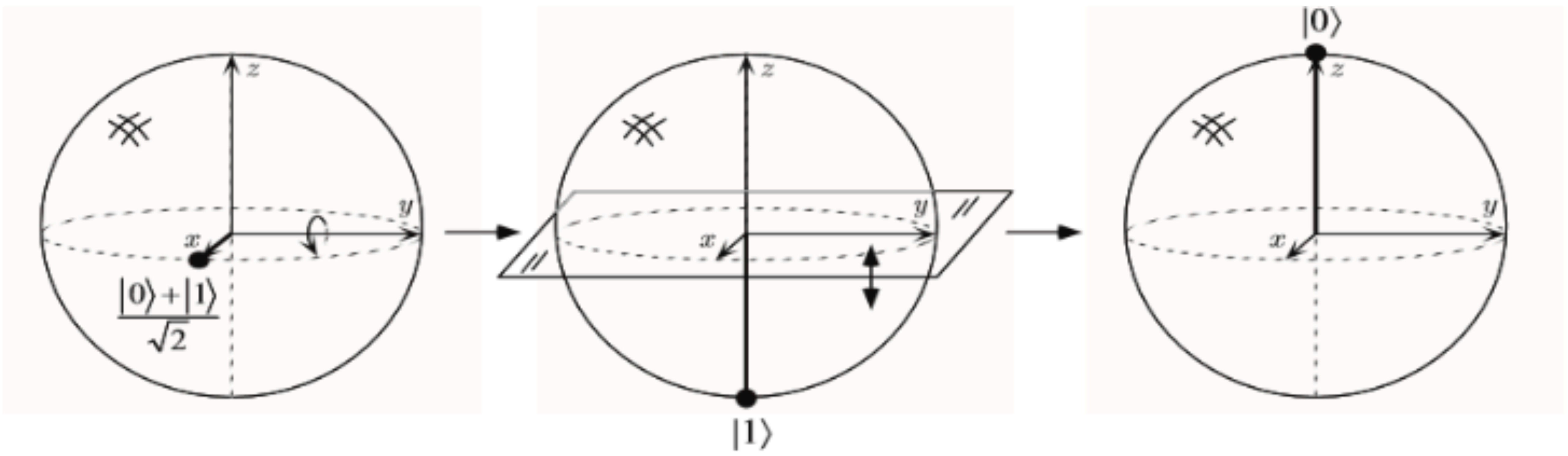


$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

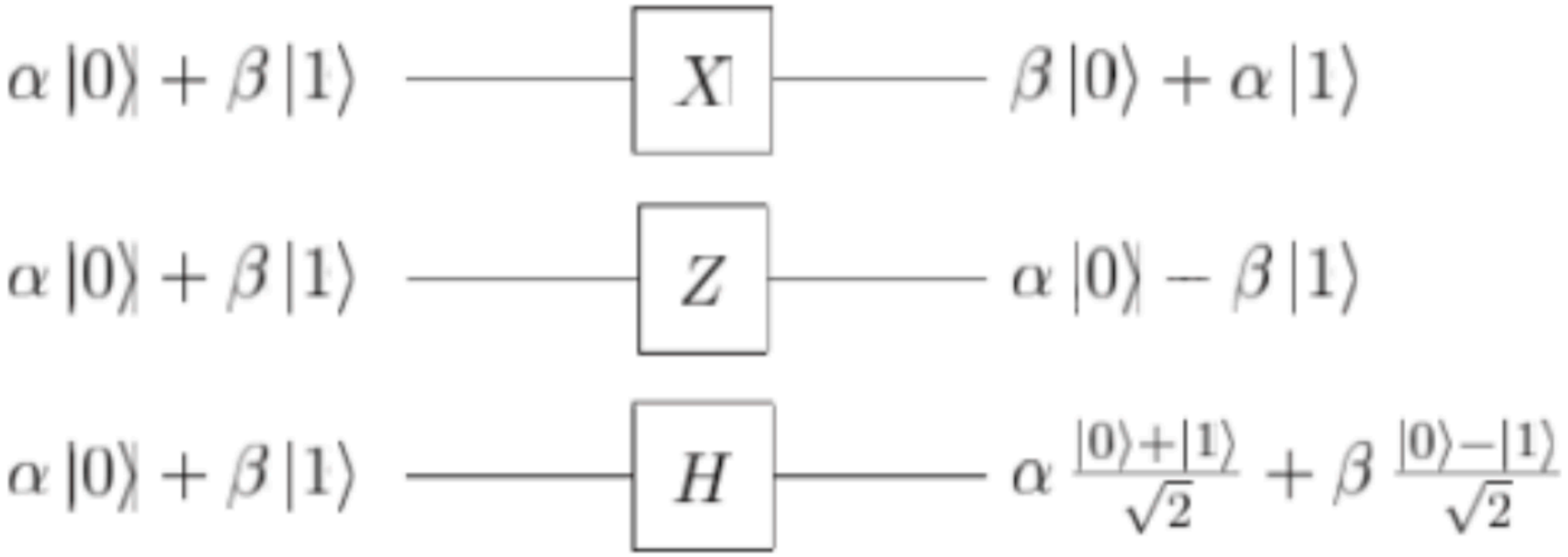


Hadamard Gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle,$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle.$$

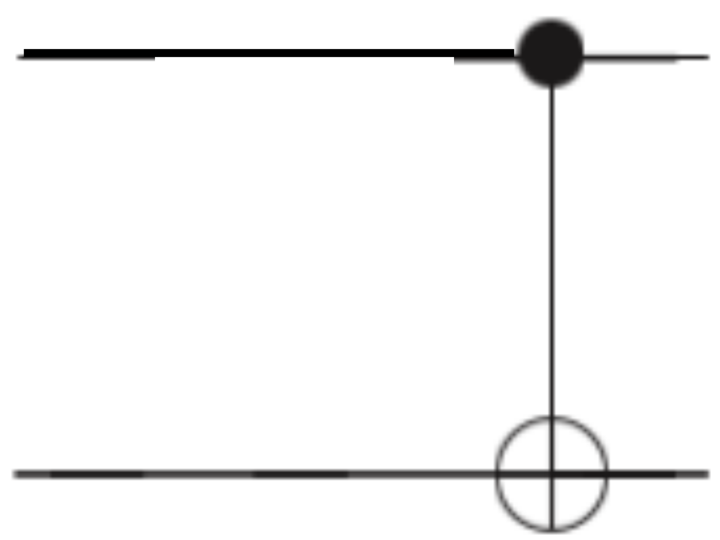


, $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$,

$|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

مروری بر مبانی نظری دروازه‌های کوانتومی

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$|\psi\rangle := \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}; \quad |\alpha|^2 + |\beta|^2 = 1.$$

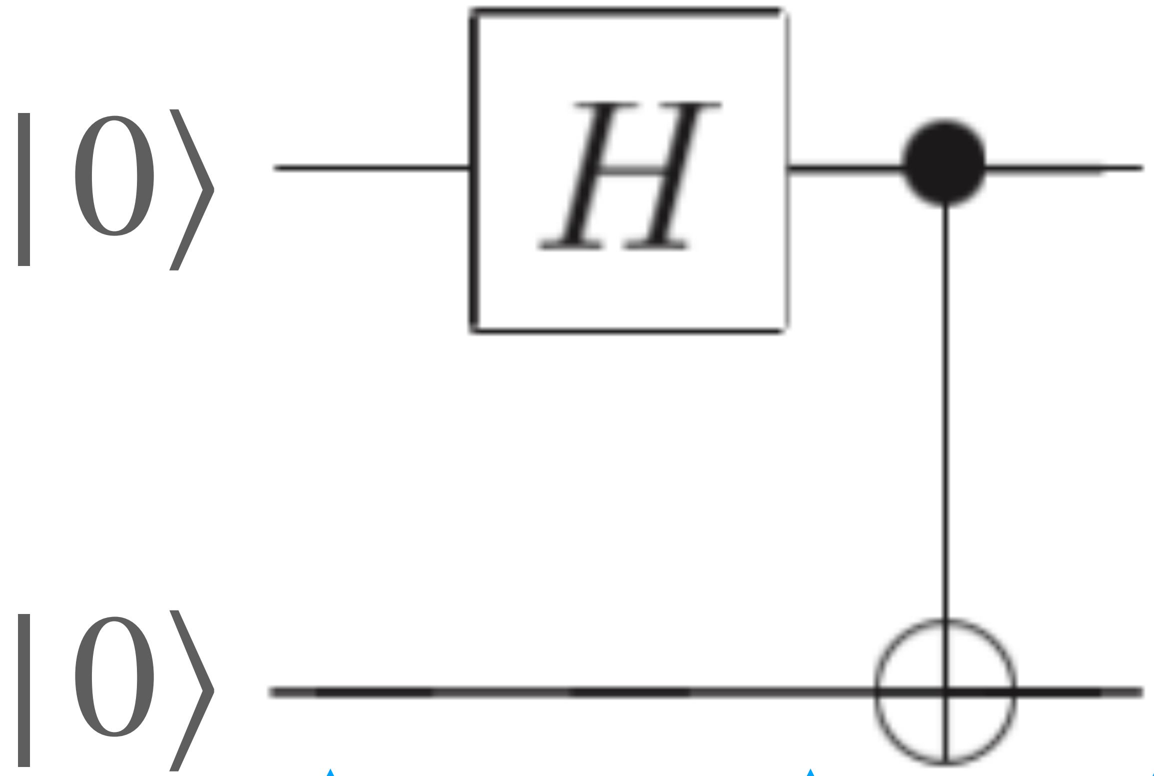
$$|00\rangle := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad |01\rangle := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad |10\rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad |11\rangle := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$CNOT\,\frac{|00\rangle + |10\rangle}{\sqrt{2}} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$CNOT|00\rangle = |00\rangle,\ CNOT|01\rangle = |01\rangle,\ CNOT|10\rangle = |11\rangle, \text{ and } CNOT|11\rangle = |10\rangle$$

مروری بر مبانی نظری دروازه‌های کوانتومی



Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|0\rangle \otimes |0\rangle = |00\rangle$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Bell state base:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

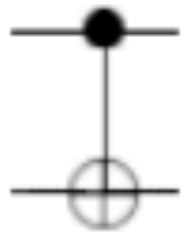
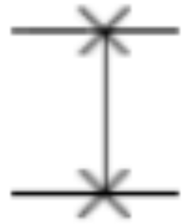
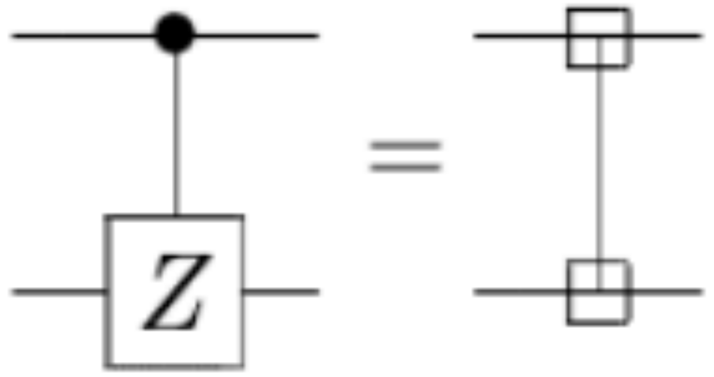
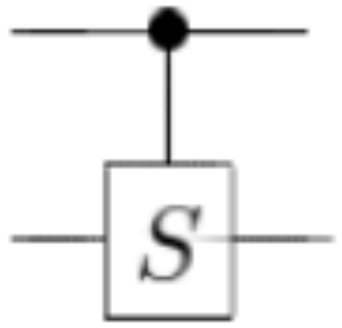
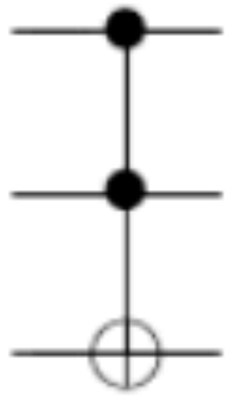
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Quantum Gates:

Gate	Symbol	Matrix
Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Pauli-X	X	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y	Y	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z	Z	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Phase	S	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$	T	$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

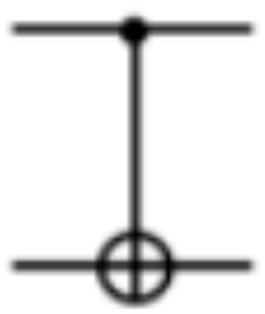
Gate	Symbol	Matrix
controlled-NOT		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
swap		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
controlled-Z		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
controlled-phase		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$
Toffoli		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Quantum Gate sets

- Clifford Gate Set
  - Hadamard gate

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

  - S gate $\begin{bmatrix}1 & 0\\0 & i\end{bmatrix}$
  - CNOT gate



$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}$$

Universal Gate Set

Hadamard gate

T gate $\begin{bmatrix}1 & 0\\0 & e^{i\pi/4}\end{bmatrix}$

CNOT gate

مروری بر مبانی نظری دروازه‌های کوانتومی

ماتریس چگالی (Density Matrix):

حالت خالص (pure state):

$$|\psi\rangle \qquad \rho = |\psi\rangle\langle\psi|$$

حالت مخلوط (mixed state):

$$\left.\begin{array}{l} p_1 : |\psi_1\rangle \\ p_2 : |\psi_2\rangle \end{array}\right\} \quad \rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$$

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \qquad tr(\rho) = \sum_i p_i = 1$$

$tr(\rho^2) = 1$ **pure state**

$tr(\rho^2) < 1$ **mixed state**

Eigenvalues of ρ ?

pure state: $$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad \rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

mixed state: $$\left.\begin{array}{l} p_1 = 1/2 : |0\rangle \\ p_2 = 1/2 : |1\rangle \end{array}\right\} \quad \rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \rho^2 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

مروری بر مبانی نظری دروازه‌های کوانتومی

اندازه‌گیری (Measurement)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\rho = |\psi\rangle\langle\psi| = \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)$$

$$= |\alpha|^2|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \xrightarrow[\text{اندازه‌گیری}]{} \rho = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

Some Terminologies

فضای هیلبرت (Hilbert space)

فضای لیوویل یا فضای هیلبرت-اشمیت (Liouville space or Hilbert-Schmidt space)

مخلوط محدب (Convex mixture)

$$\rho = p\rho_\psi + (1-p)\rho_\phi$$

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

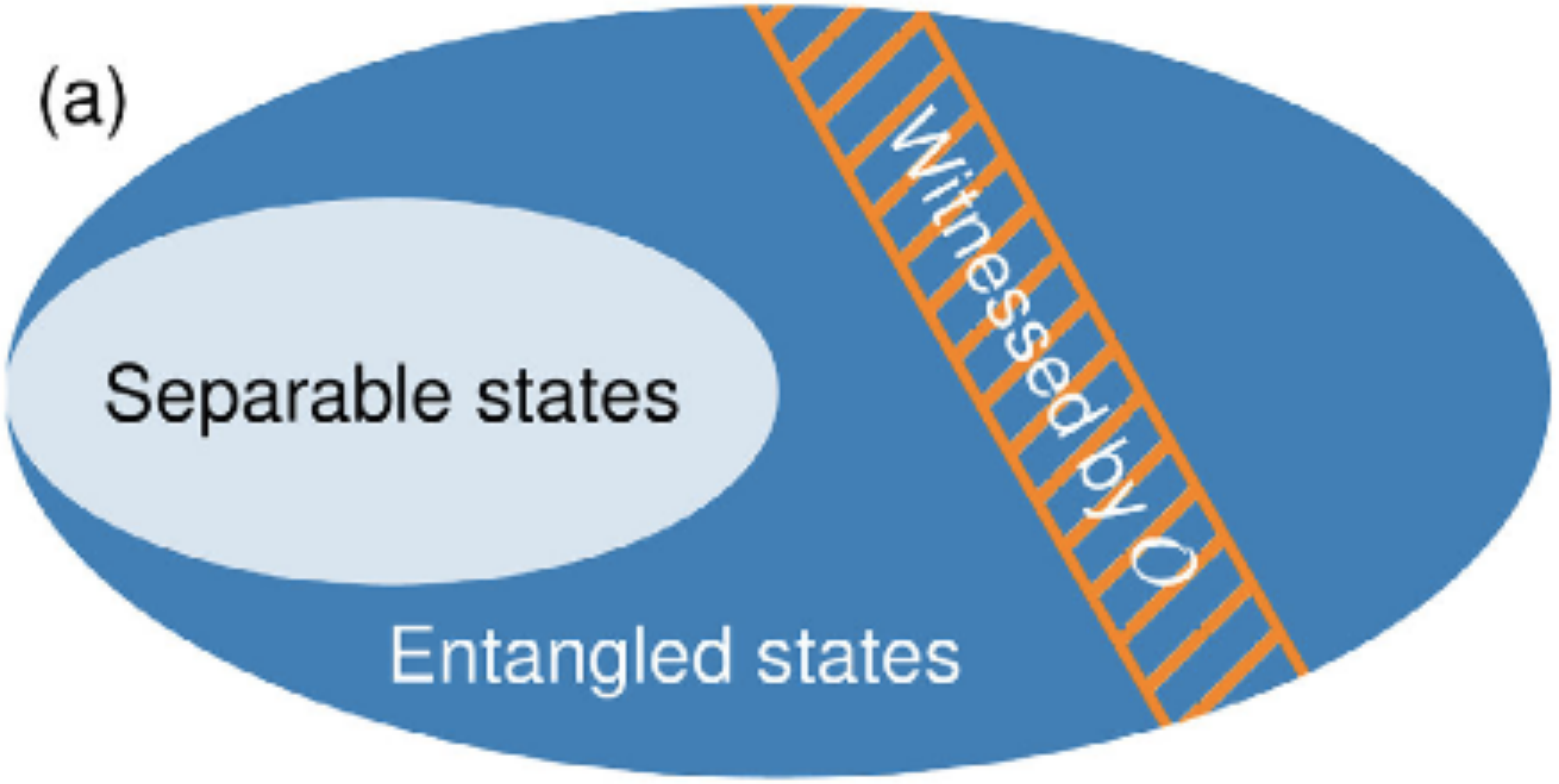
$$\frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$$

حالتهای خالص (Pure states)

حالتهای مخلوط (Mixed states)

حالتهای درهمتنیده (Entangled states)

حالتهای تفکیک پذیر (Separable states) —> convex set



مروری بر مبانی نظری دروازه‌های کوانتومی

ماتریس چگالی کاهش یافته (Reduced Density Matrix):

state space is the tensor product $H_A \otimes H_B$ of Hilbert spaces

$$\rho^A = \mathrm{Tr}_B\,\rho$$

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle) \qquad \rho = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$\rho^A = Tr_B\rho = \langle 0|_B\,\rho\,|0\rangle_B + \langle 1|_B\,\rho\,|1\rangle_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

اطلاعات کلاسیک و کوانتومی

- آنتروپی شانون
  (Shannon Entropy)

$$S = -\sum_i P_i \log_2 P_i$$

- آنتروپی بولتزمن
  (Boltzman Entropy)

$$S = -k_B \sum_i P_i \ln P_i$$

- آنتروپی وون نیومن
  (von Neumann Entropy)

$$S = -Tr(\rho \log_2 \rho)$$

(Shannon Entropy) آنتروپی شانون

1	P(1) = 1/2	0
2	P(2) = 1/4	10
3	P(3) = 1/8	110
4	P(4) = 1/8	111

$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\frac{1}{8}\cdot 3 = 7/4 \text{ bits}$$

$$S = -\sum_i P_i \log_2 P_i = -1/2\log(1/2) - 1/4\log(1/4) - 1/8\log(1/8) - 1/8\log(1/8) = 7/4$$

آنتروپی درهمتنیدگی (Entropy of Entanglement)

🔷 What is the Entropy of Entanglement?

The **entropy of entanglement** is a way of quantifying **how entangled** a pure bipartite quantum state is.

Suppose we have a bipartite pure state shared between subsystems A and B:

$$|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

1. Compute the reduced density matrix of one subsystem (say A):

$$\rho_A = \mathrm{Tr}_B\big(|\psi\rangle\langle\psi|\big)$$

2. The **entropy of entanglement** is defined as the **von Neumann entropy** of this reduced density matrix:

$$E(|\psi\rangle) = S(\rho_A) = -\mathrm{Tr}\big(\rho_A \log_2 \rho_A\big)$$

Because the global state is pure, the entropy of subsystem A equals that of B.

آنتروپی درهمتنیدگی (Entropy of Entanglement)

🔷 Example 1: A Bell State

Take the maximally entangled Bell state:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Full density matrix:

$$\rho = |\Phi^+\rangle\langle\Phi^+|$$

- Reduced density matrix for subsystem A:

$$\rho_A = \mathrm{Tr}_B(\rho) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I$$

- This is a **maximally mixed state**.
- Entropy:

$$E(|\Phi^+\rangle) = -\left(\tfrac{1}{2}\log_2 \tfrac{1}{2} + \tfrac{1}{2}\log_2 \tfrac{1}{2}\right) = 1$$

✅ The entropy of entanglement is **1 bit** — maximal for two qubits.

🔷 Example 2: A Partially Entangled State

$$|\psi\rangle = \sqrt{0.9}\,|00\rangle + \sqrt{0.1}\,|11\rangle$$

- Reduced density matrix of A:

$$\rho_A = 0.9|0\rangle\langle 0| + 0.1|1\rangle\langle 1|$$

- Entropy:

$$E(|\psi\rangle) = -(0.9\log_2 0.9 + 0.1\log_2 0.1) \approx 0.469 \text{ bits}$$

So, less than 1 — the entanglement is **not maximal**.

Partial Matrix Transposition and Time Reversal

$$|\Phi\rangle_{AB} = 1/\sqrt{2}(|00\rangle + |11\rangle)$$

$$\rho_{AB} = |\Phi\rangle\langle\Phi| = 1/2\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_{1,2,3,4} = 1,\ 0,\ 0,\ 0$$

$$\left(I_A \otimes T\right)\rho_{AB} = |\Phi\rangle\langle\Phi| = 1/2\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

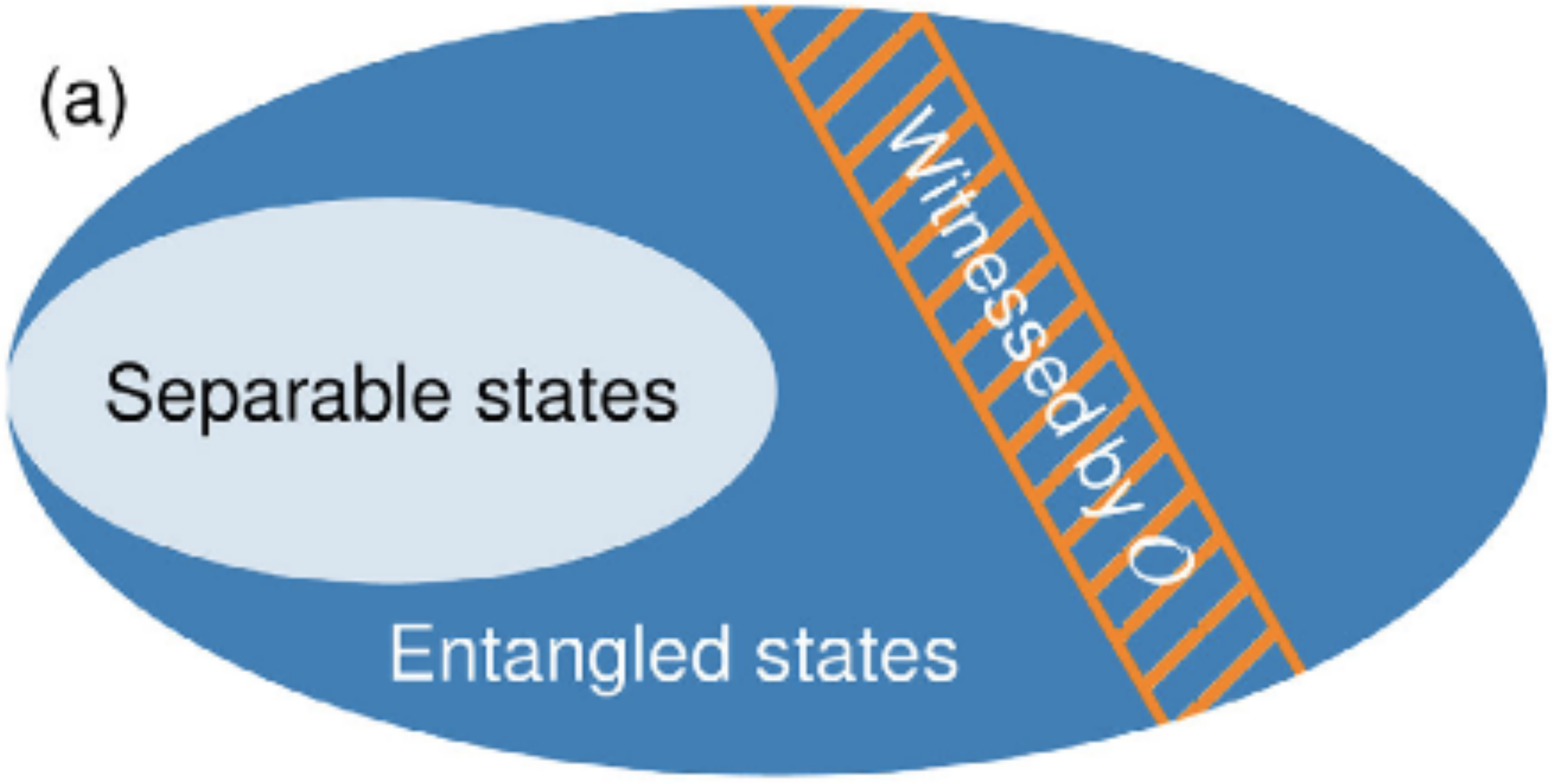
$$(I_A \otimes T)(|\Phi\rangle\langle\Phi|) = \left(I_A \otimes T\right)\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_{1,2,3,4} = \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}$$

Entanglement Witness (EW)

$\mathrm{Tr}(W\rho_{\mathrm{sep}}) \geq 0$ for all separable states ρ_{sep}.

$\mathrm{Tr}(W\rho_{\mathrm{ent}}) < 0$ for at least one entangled state ρ_{ent}.



$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad \rho = |\psi^-\rangle\langle\psi^-|$$

$$W = \tfrac{1}{2}I - |\psi^-\rangle\langle\psi^-|$$

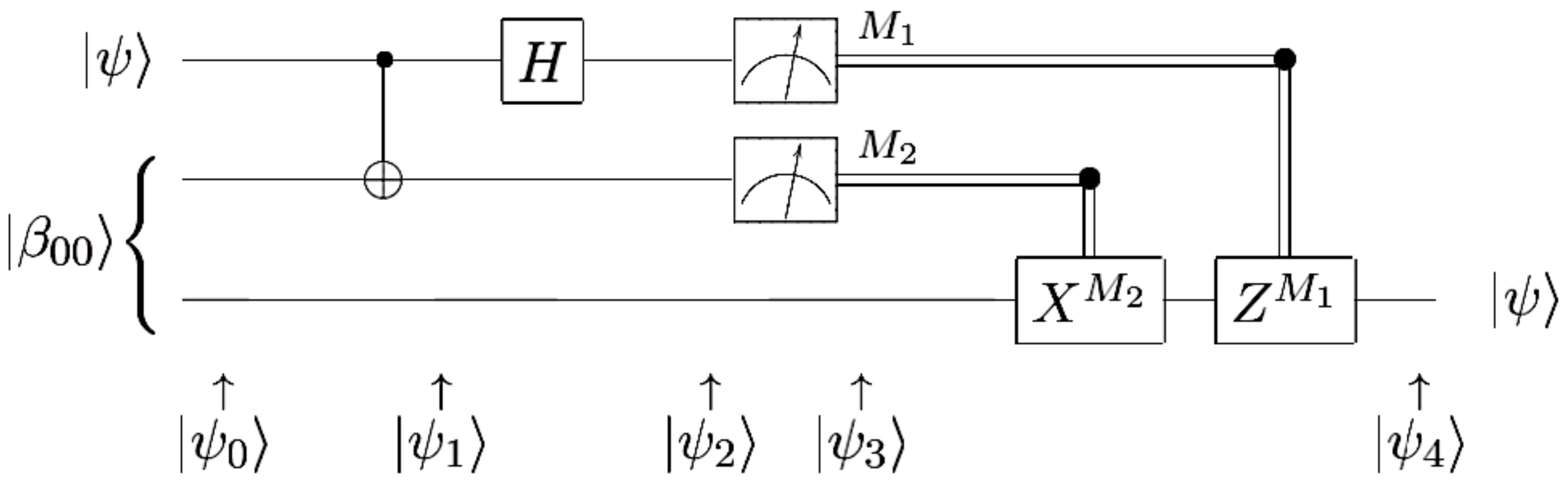
$$|\langle\psi^-|(a \otimes b)\rangle|^2 \leq \frac{1}{2}$$

$$\langle\psi^-|W|\psi^-\rangle = \tfrac{1}{2} - 1 = -\tfrac{1}{2} < 0$$

$$\langle W\rangle = \tfrac{1}{2} - |\langle\psi^-|a \otimes b\rangle|^2 \geq \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$\mathrm{Tr}(W\rho) = \tfrac{1}{2} - 1 = -\tfrac{1}{2} < 0$$

دوربری کوانتومی (Quantum Teleportation)



$$|\psi_2\rangle = \frac{1}{2}\Big[|00\rangle\,(\alpha|0\rangle + \beta|1\rangle) + |01\rangle\,(\alpha|1\rangle + \beta|0\rangle) + |10\rangle\,(\alpha|0\rangle - \beta|1\rangle) + |11\rangle\,(\alpha|1\rangle - \beta|0\rangle)\Big]$$

$$00 \longmapsto |\psi_3(00)\rangle \equiv \Big[\alpha|0\rangle + \beta|1\rangle\Big]$$

$$01 \longmapsto |\psi_3(01)\rangle \equiv \Big[\alpha|1\rangle + \beta|0\rangle\Big]$$

$$10 \longmapsto |\psi_3(10)\rangle \equiv \Big[\alpha|0\rangle - \beta|1\rangle\Big]$$

$$11 \longmapsto |\psi_3(11)\rangle \equiv \Big[\alpha|1\rangle - \beta|0\rangle\Big].$$

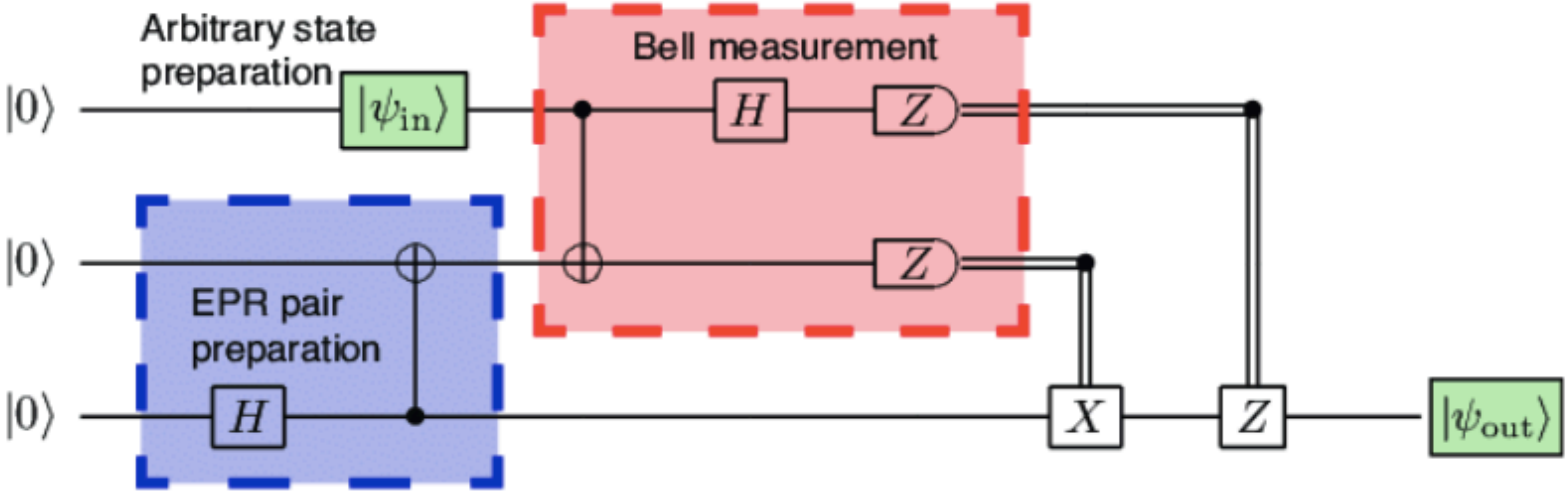
$$|\psi\rangle = a|0\rangle + b|1\rangle \qquad |\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\Big]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\Big]$$

$$|\psi_2\rangle = \frac{1}{2}\Big[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\Big]$$

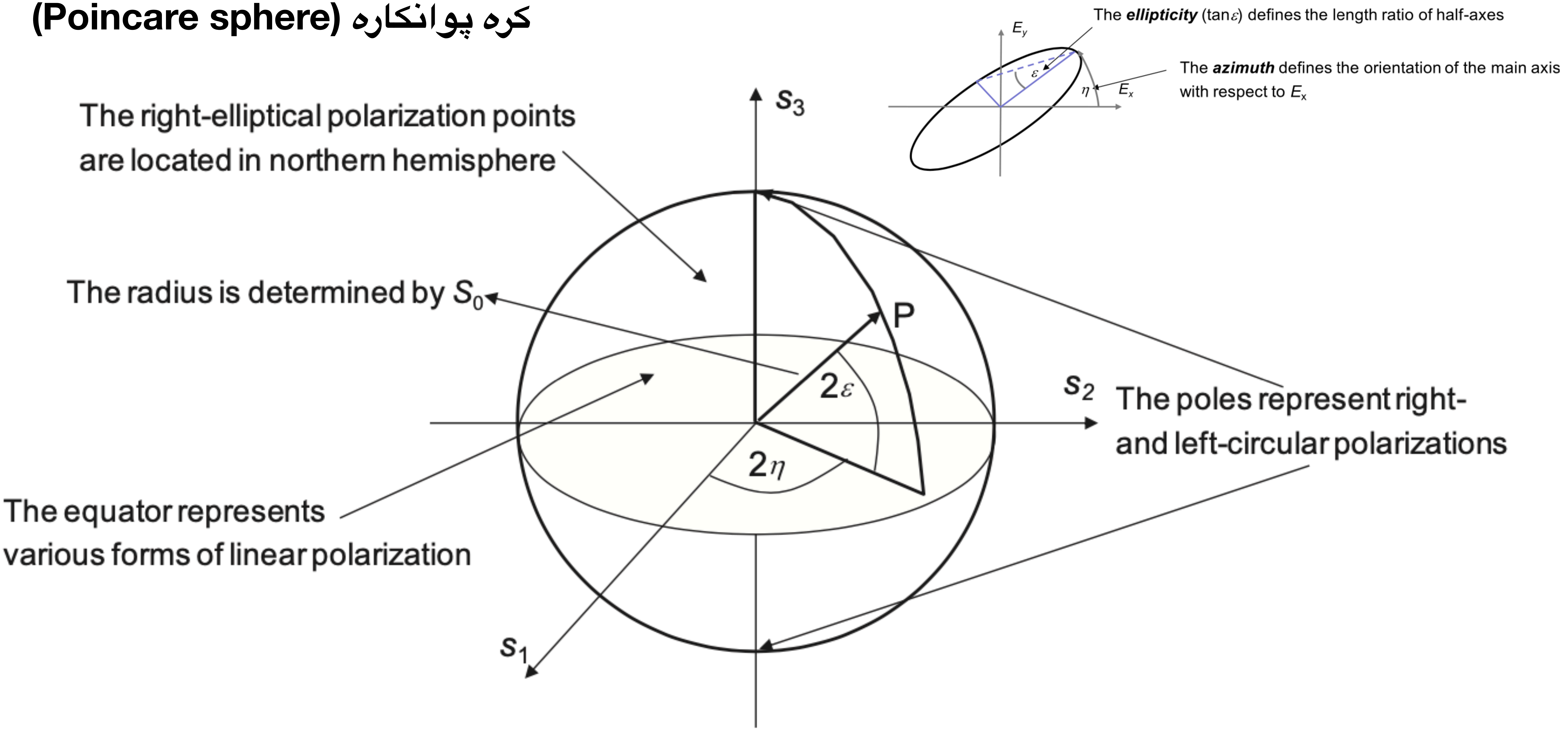
Nielsen, M. A., & Chuang, I. L. (2010). *Quantum computation and quantum information*. Cambridge university press.



Arbitrary state preparation
Bell measurement
EPR pair preparation
|0⟩
|0⟩
|0⟩
|ψin⟩
H
Z
Z
H
X
Z
|ψout⟩

قطبش تک فوتون بعنوان یک کیوبیت

کره پوانکاره (Poincare sphere)

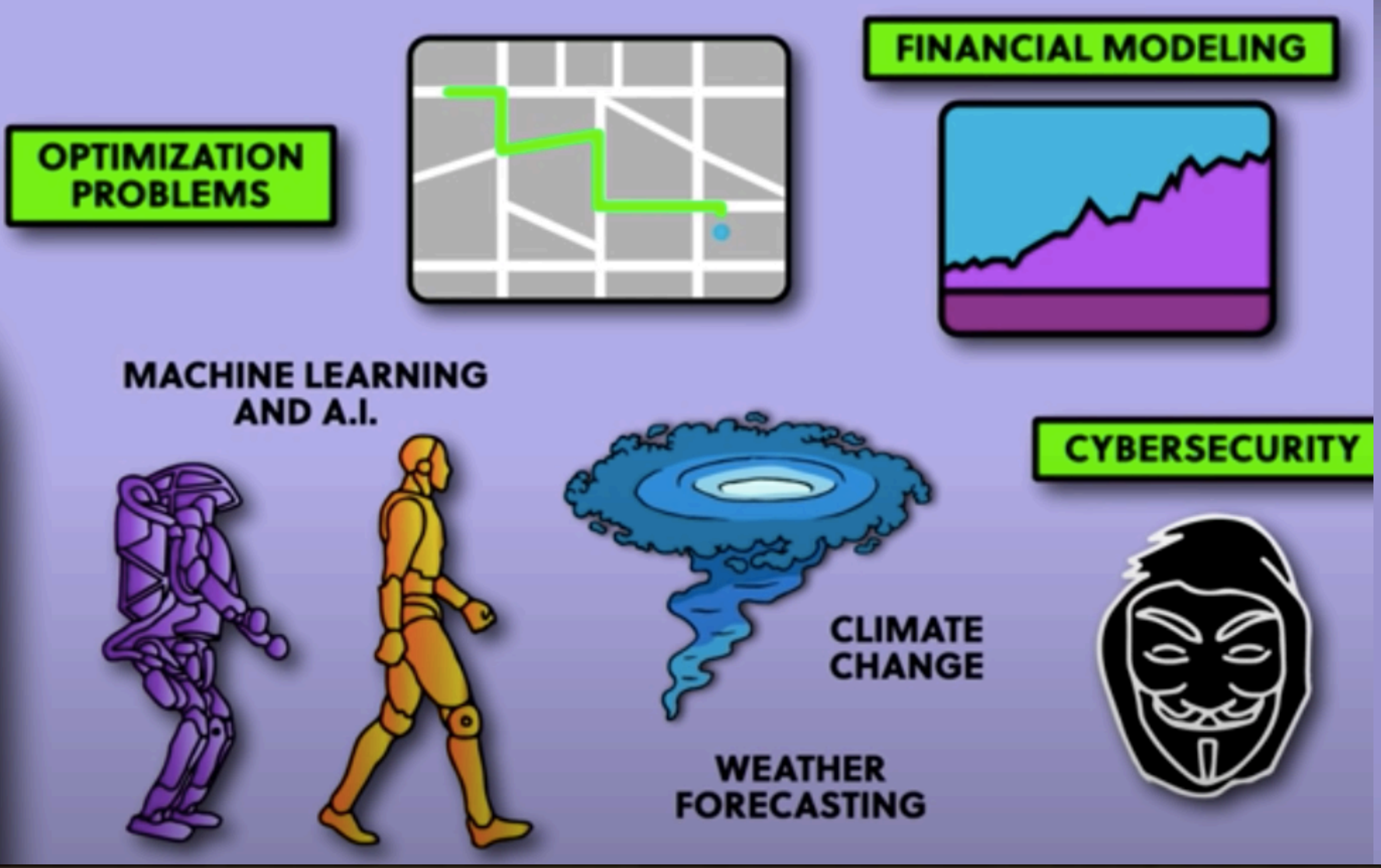


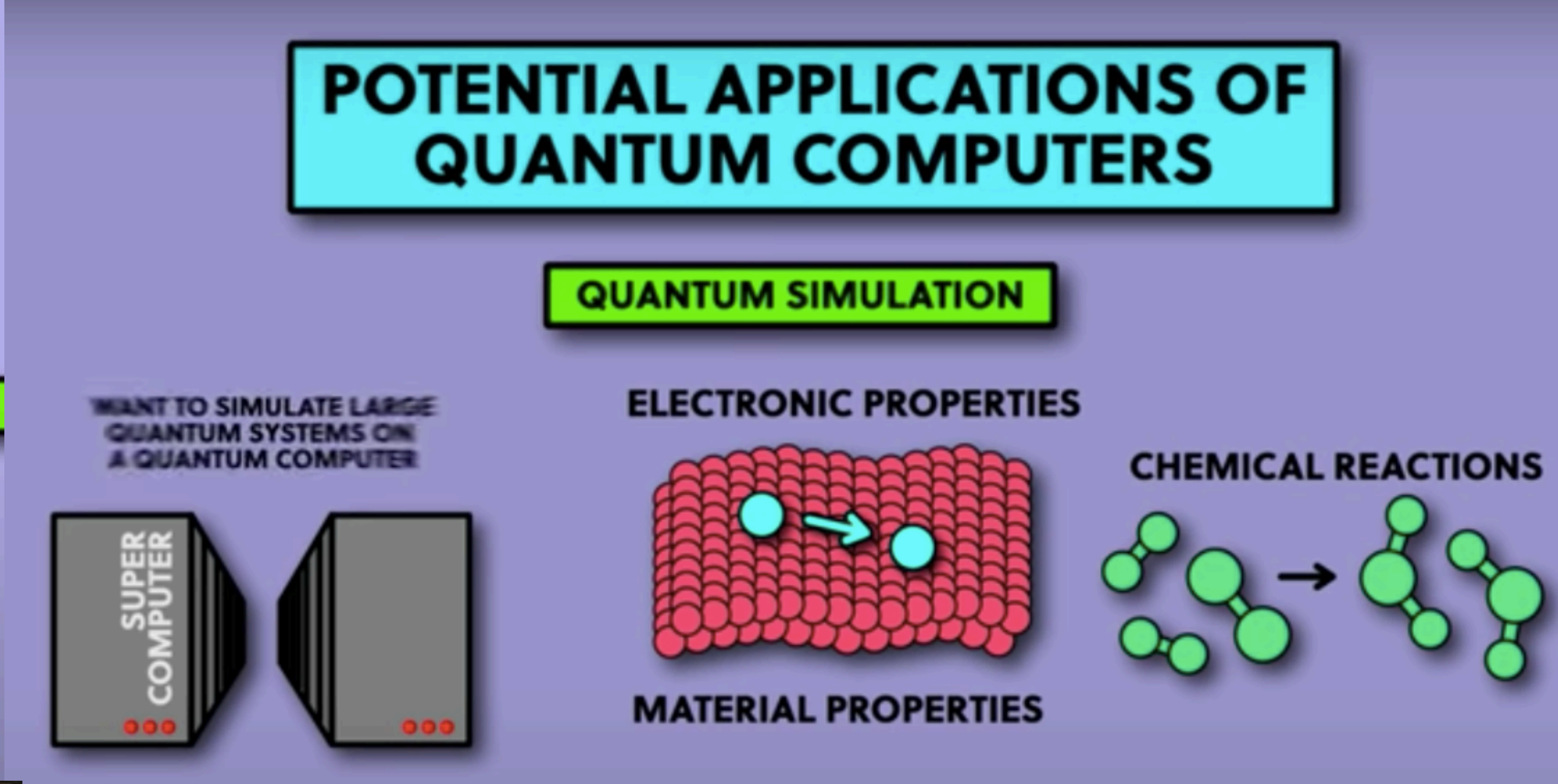
Quantum Key Distribution (QKD)

BB84 protocol: Charles H. Bennett and Gilles Brassard (1984)

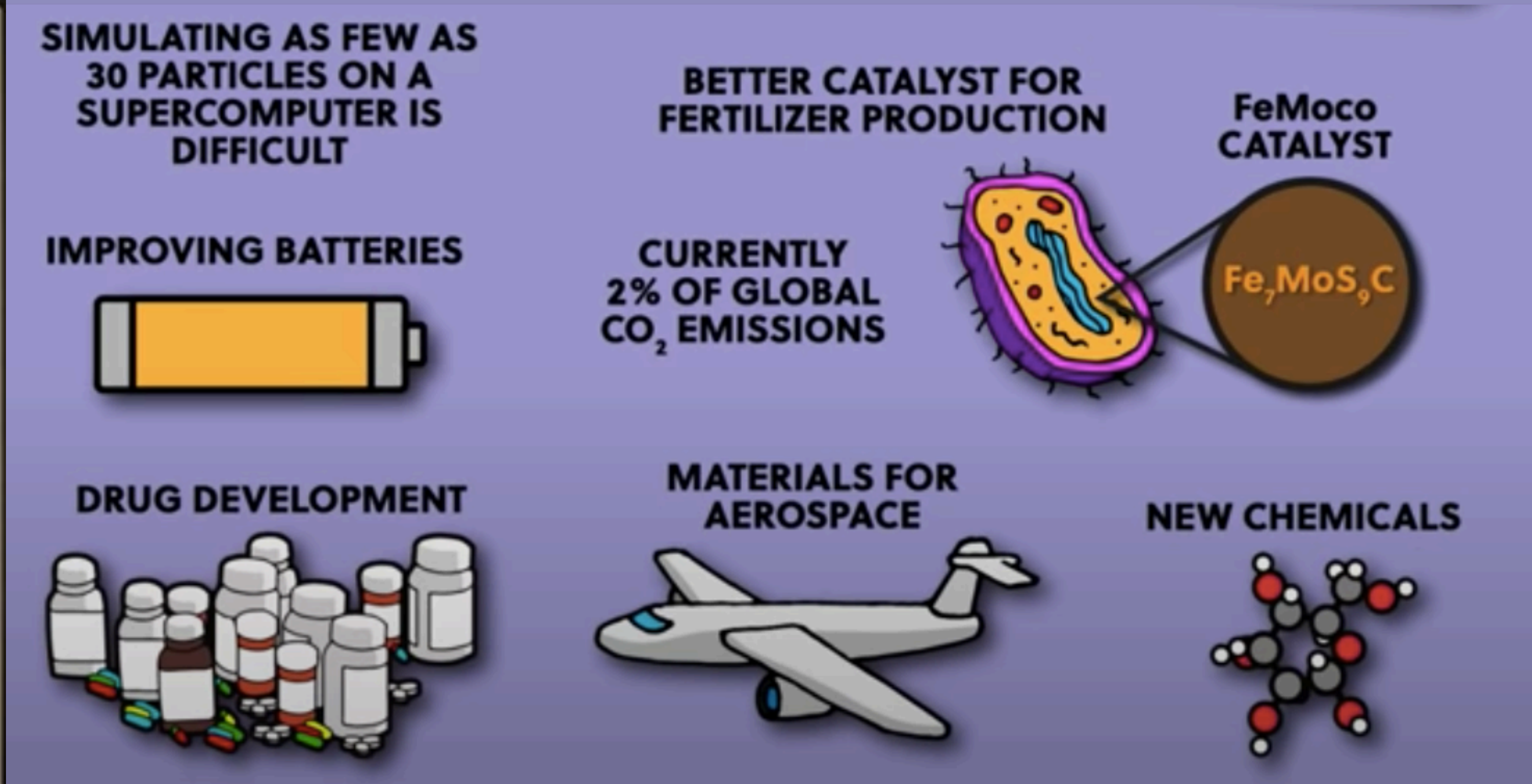
Basis	0	1
+	↑	→
×	↗	↘

Alice's random bit	0	1	1	0	1	0	0	1
Alice's random sending basis	+	+	×	+	×	×	×	+
Photon polarization Alice sends	↑	→	↘	↑	↘	↗	↗	→
Bob's random measuring basis	+	×	×	×	+	×	+	+
Photon polarization Bob measures	↑	↗	↘	↗	→	↗	→	→
PUBLIC DISCUSSION OF BASIS								
Shared secret key	0		1			0		1





با تشکر از توجه شما



قوی ترین مجموعه‌های کامپیوترهای کوانتومی جهان

IBM Q System One (Fraunhofer)

1- IBM ——> QASM , Qiskit platforms

2- Google ——> Tensorflow Quantum (TFQ) , Cirq

3- ETH Zurich ——> ProjectQ

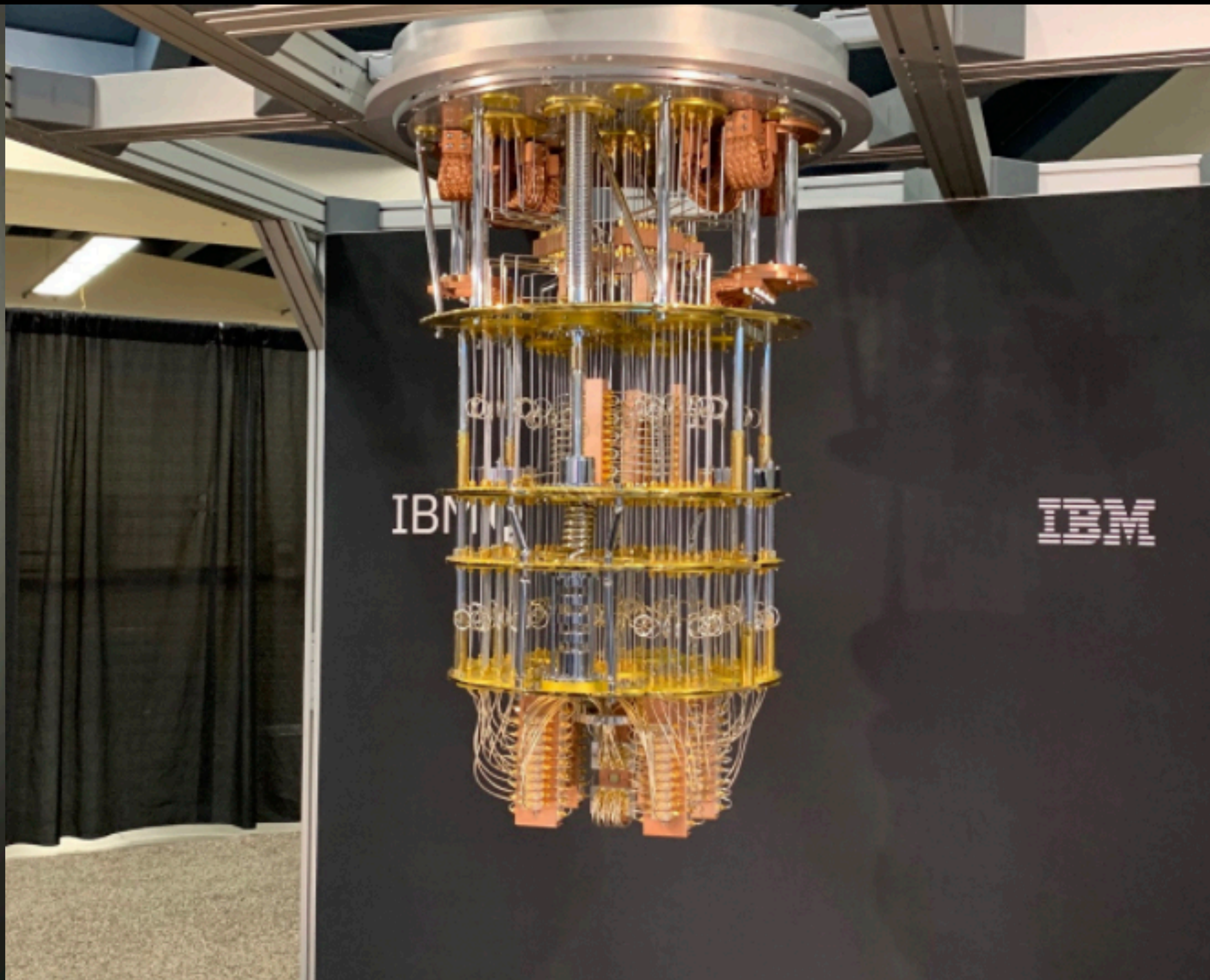
4- Perceval (France) ——> photonic quantum computer

5- Microsoft ——> Q#

6- Xanadu (Toronto) ——> Strawberry Field platform ——> Pennylane

7- NASA Google ——> D-wave Quantum Annealer

IBM Quantum Computer

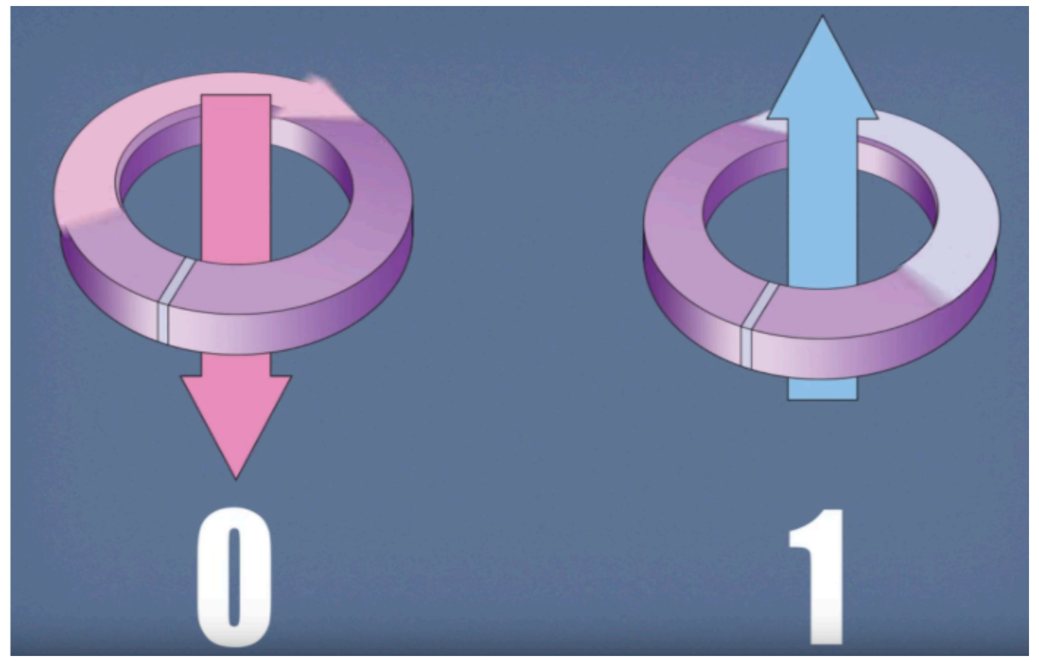


D-Wave System

For the D-Wave system, the Hamiltonian may be represented as

$$\mathcal{H}_{ising} = \underbrace{-\frac{A(s)}{2}\left(\sum_i \hat{\sigma}_x^{(i)}\right)}_{\text{Initial Hamiltonian}} + \underbrace{\frac{B(s)}{2}\left(\sum_i h_i \hat{\sigma}_z^{(i)} + \sum_{i>j} J_{i,j} \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)}\right)}_{\text{Final Hamiltonian}}$$

where $\hat{\sigma}_{x,z}^{(i)}$ are Pauli matrices operating on a qubit q_i, and h_i and $J_{i,j}$ are the qubit biases and coupling strengths.[1]



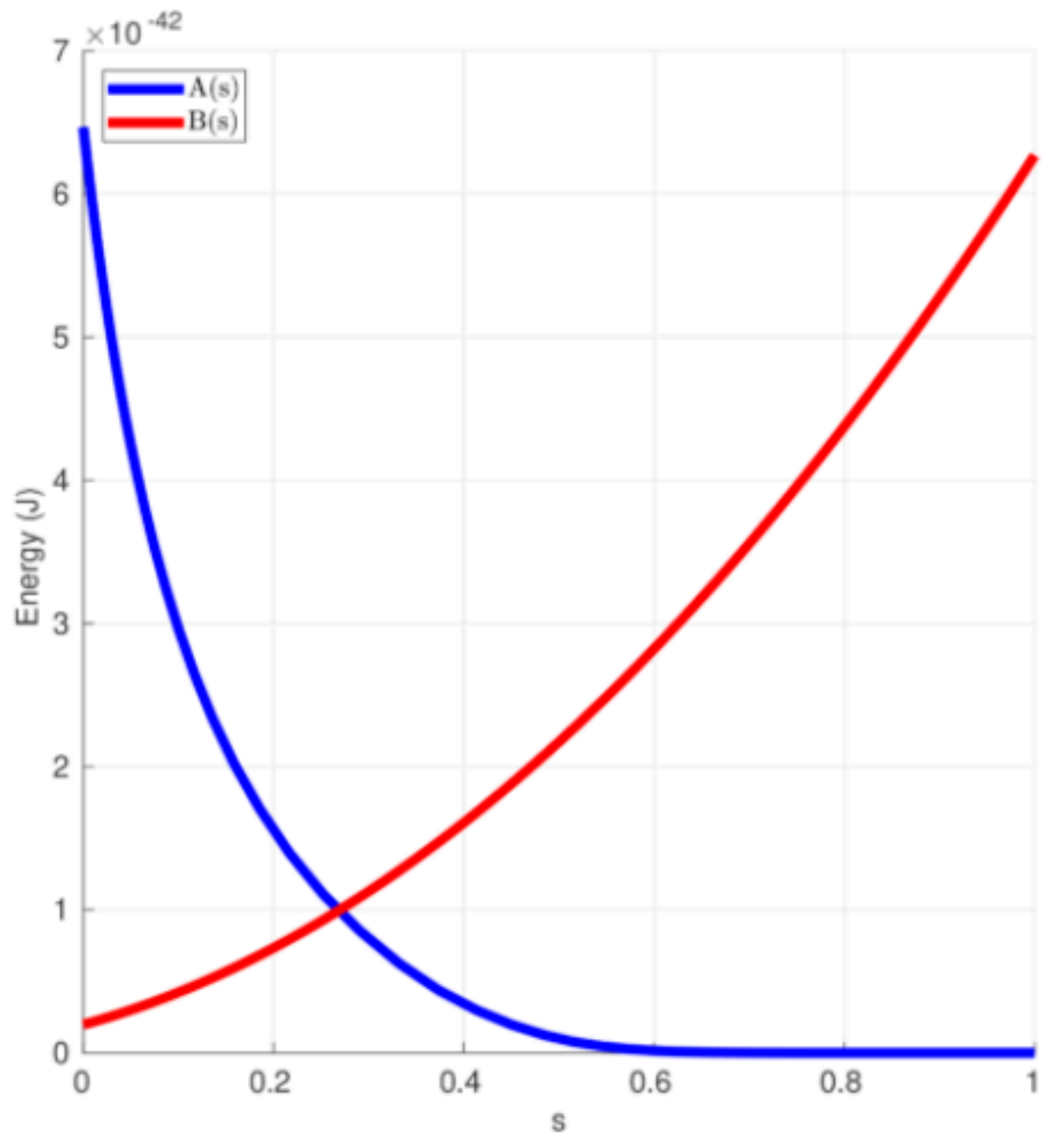
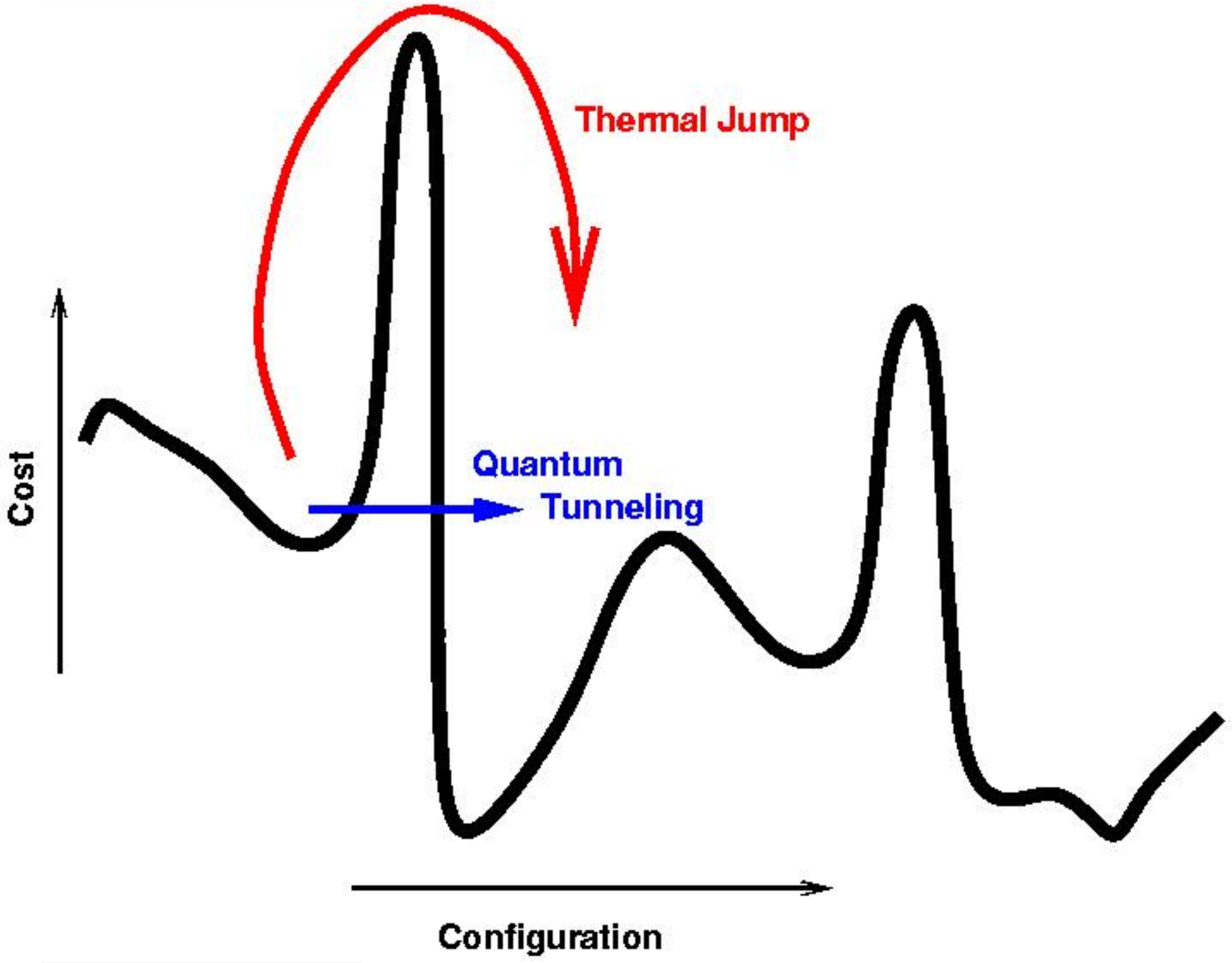


Fig. 8 Annealing functions $A(s)$, $B(s)$. Annealing begins at $s = 0$ with $A(s) \gg B(s)$ and ends at $s = 1$ with $A(s) \ll B(s)$. Data shown are representative of D-Wave 2X systems.



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