Non-Abelian duality symmetry in the WZW model on the $SO(2,2)$ Lie group

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The first superstring revolution

- The first superstring revolution began in 1984 with the discovery that quantum mechanical consistency of a ten-dimensional theory with $N = 1$ supersymmetry requires a local Yang-Mills gauge symmetry based on one of two possible Lie algebras: $SO(32)$ or $E_8 \times E_8$. Only for these two choices do certain quantum mechanical anomalies cancel.

- When one uses the superstring formalism for both left-moving modes and right-moving modes, the supersymmetries associated with the left-movers and the right-movers can have either opposite handedness or the same handedness. These two possibilities give different theories called the **type IIA** and **type IIB** superstring theories, respectively.

The first superstring revolution

- A third possibility, called **type I** superstring theory, can be derived from the type IIB theory by modding out by its left-right symmetry, a procedure called orientifold projection.

- Altogether, there are five distinct superstring theories, each in ten dimensions. Three of them, the type I theory and the two heterotic theories, have $N = 1$ supersymmetry in the ten-dimensional sense. The minimal spinor in ten dimensions has 16 real components, so these theories have 16 conserved supercharges. The type I superstring theory has the gauge group $SO(32)$. The other two theories, type IIA and type IIB, have $N = 2$ supersymmetry or equivalently 32 supercharges.

The first superstring revolution

- In the late 1980s it was realized that there is a property known as Target space duality **(T-duality)** that relates the two type II theories and the two heterotic theories, so that they shouldn't really be regarded as distinct theories.

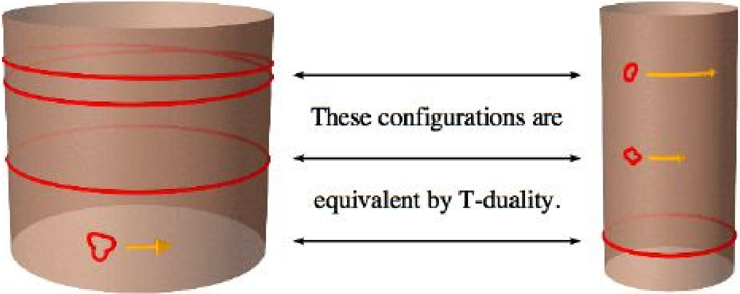
- T-duality implies that in many cases two different geometries for the extra dimensions are physically equivalent! In the simplest example, a circle of radius R is equivalent to a circle of radius l_s^2/R, where l_s^2 is the fundamental string length scale. T-duality typically relates two different theories. For example, it relates the two type II and the two heterotic theories. Therefore, the type IIA and type IIB theories (also the two heterotic theories) should be regarded as a single theory.

T-duality

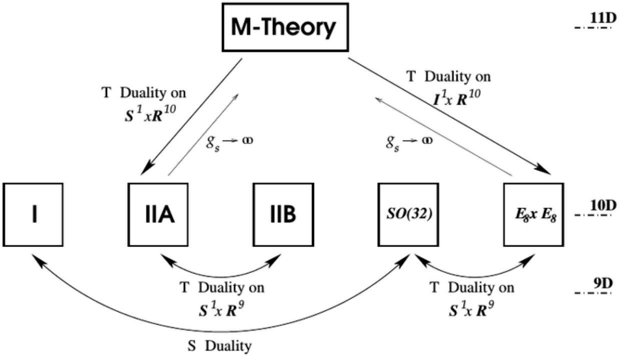
T-Duality

That surprising symmetry is very important

- Everything matches, not just the energy spectrum!
  - Honestly *no difference* between universes with R and l_s^2 / R



The various duality transformations relating the superstring theories.



Introduction to duality symmetries in string theory

- T-duality connects seemingly different backgrounds in which the strings can propagate. String backgrounds and their dual fields can be considered as different descriptions of the same physical system. In other words, they represent the same point in moduli space of a given string theory.

- Duality symmetries were originally discovered for toroidal compactifications of closed string theories.
  [K. Kikkawa and M. Yamasaki, **Phys. Lett.** 149B (1984) 357]
  [N. Sakai and I. Senda, **Progr. Theor. Phys. Suppl.** 75 (1986) 692]

Introduction to duality symmetries in string theory

- Buscher generalized the $R \leftrightarrow \frac{\alpha'}{R}$ duality in toroidal compactifications of string theories to any string background for which the metric in the worldsheet action had at least one isometry.
  [T. H. Buscher, **Phys. Lett. B**194 (1987) 59]
  [T. H. Buscher, **Phys. Lett. B**201 (1988) 466]

- Unlike S-duality [Seiberg and Witten (1994)], the T-duality transformations are formulated at the level of the two-dimensional non-linear sigma model. There are, however, no general methods for constructing these transformations.

- **Abelian T-duality**:
  The duality is Abelian if it is constructed on an Abelian isometry group. We refer to Abelian T-duality for stressing the presence of global Abelian isometries in the target spaces of both the paired sigma models.
  [T. H. Buscher, **Phys. Lett. B**194 (1987) 59]
  [T. H. Buscher, **Phys. Lett. B**201 (1988) 466]

  An important feature of the Abelian case is the possibility to reverse this procedure and obtain back the original model from the dual one. This is due to the fact that Abelian T-duality preserves the symmetries of the original sigma model.

- **Non-Abelian T-duality**:
  The non-Abelian T-duality refers to the existence of a non-Abelian isometry on the target space of one of the two σ-models.
  [X. C. de la Ossa and F. Quevedo, **Nucl. Phys. B** 403, 377 (1993), arxiv:hep-th/9210021.]

  However, the symmetries of the original theory are not preserved and non-Abelian T-duality is not reversible.
  [A. Giveon and M. Rocek, **Nucl. Phys. B** 421, 173 (1994), arxiv:hep-th/9308154.]

Poisson-Lie T-duality

- **Poisson-Lie T-duality**:
  Poisson-Lie T-duality, a generalization of T-duality, does not require existence of isometry in the original target manifold; the integrability of the Noether's currents associated with the action of group G on the target manifold is enough to have this symmetry. In other words, the components of the Noether's currents play the role of flat connection, i.e. they satisfy Maurer-Cartan equations with group structure of $\tilde{G}$ (with the same dimension of G), so that G and $\tilde{G}$ have Poisson-Lie structure and their Lie algebras form a Lie bialgebra.

  [C. Klimcik and P. Severa, **Phys. Lett. B** 351, 455 (1995), arxiv:hep-th/9502122]
  [C. Klimcik, **Nucl. Phys. Proc. Suppl.** 46, 116 (1996), arxiv:hep-th/9509095]

Poisson-Lie symmetry on manifolds

Two-dimensional sigma model

$$S = -\frac{T}{2}\int d\tau\, d\sigma \left(\sqrt{-h}h^{\alpha\beta}\, G_{\mu\nu}(x)\, \partial_\alpha x^\mu \partial_\beta x^\nu + \epsilon^{\alpha\beta}\, B_{\mu\nu}(x)\, \partial_\alpha x^\mu \partial_\beta x^\nu\right), \qquad (0.1)$$

- $\alpha = \tau, \sigma, \qquad h = \det(h_{\alpha\beta})$
- $x^\mu, \quad \mu = 0, 1, ..., d-1,$

The action in light-cone coordinates

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^-\, \mathcal{E}_{\mu\nu}(x)\, \partial_+ x^\mu \partial_- x^\nu, \qquad (0.2)$$

- $\sigma^\pm = \frac{\tau\pm\sigma}{2}, \quad \mathcal{E}_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$

Poisson-Lie symmetry on manifolds

- The right action of the Lie group G on the target $\mathcal{M}$

$$V_a = V_a{}^{\mu} \frac{\partial}{\partial x^{\mu}} , \qquad (a = 1, ..., dimG), \tag{0.3}$$

- Variation of action under transformation

$$x^{\mu} \longrightarrow x^{\mu} + \epsilon^a(\sigma^+, \sigma^-) \; V_a{}^{\mu}, \tag{0.4}$$

$$\begin{aligned} \delta S = \delta S(x + \epsilon^a \; V_a) = \frac{1}{2} \int d\sigma^+ d\sigma^- \; (\mathcal{L}_{V_a} \mathcal{E}_{\mu\nu}) \; \partial_+ x^{\mu} \partial_- x^{\nu} \\ - \frac{1}{2} \int d\epsilon^a \wedge \star J_a, \end{aligned} \tag{0.5}$$

Poisson-Lie symmetry on manifolds

- Lie derivative

$$\mathcal{L}_{V_a}\,\mathcal{E}_{\mu\nu} = \partial_\mu V_a{}^{\lambda}\mathcal{E}_{\lambda\nu} + V_a{}^{\lambda}\partial_\lambda\mathcal{E}_{\mu\nu} + \partial_\nu V_a{}^{\lambda}\mathcal{E}_{\mu\lambda}$$

- The Hodge star of Noether's current one-forms

$$\star J_a = V_a{}^{\mu}\,\partial_+ x^{\nu}\,\mathcal{E}_{\nu\mu}\,d\sigma^+ - V_a{}^{\mu}\,\mathcal{E}_{\mu\nu}\,\partial_- x^{\nu}\,d\sigma^-.$$

- By direct calculation, one can consider

$$d\star J_a = -\Big[(\mathcal{L}_{V_a}\,\mathcal{E}_{\mu\nu})\,\partial_+ x^{\mu}\partial_- x^{\nu} + V_a{}^{\nu}\,(\textit{equations of motion})\Big]\,d\sigma^+\wedge d\sigma^-, \quad (0.6)$$

Poisson-Lie symmetry on manifolds

- Thus, on extremal surface we have $\delta S = 0$ and

$$d\star {}_iJ = -(\mathcal{L}_{V_a}\, \mathcal{E}_{\mu\lambda})\, \partial_+ x^{\mu} \partial_- x^{\lambda}\; d\sigma^+ \wedge d\sigma^-, \tag{0.7}$$

  - If $\mathcal{L}_{V_a}\, \mathcal{E}_{\mu\lambda} = 0$, then G is a isometry group of $\mathcal{M}$
  - If $\star J_a$ on extremal surfaces satisfy Maurer-cartan equation

$$d\star J_a = -\frac{1}{2}\tilde{f}^{bc}{}_a\, \star J_b \wedge \star J_c, \tag{0.8}$$

Poisson-Lie symmetry

$$\mathcal{L}_{V_a}(\mathcal{E}_{\mu\lambda}) = \tilde{f}^{bc}{}_a\, \mathcal{E}_{\mu\rho}\; V_c{}^{\rho}\; V_b{}^{\sigma}\; \mathcal{E}_{\sigma\lambda}, \tag{0.9}$$

- Now, using integrability condition on Lie derivative

$$\mathcal{L}_{[V_a, V_b]}(\mathcal{E}_{\mu\lambda}) = [\mathcal{L}_{V_a}, \mathcal{L}_{V_b}]\mathcal{E}_{\mu\lambda} = \mathcal{L}_{V_a}\, \mathcal{L}_{V_b}\, \mathcal{E}_{\mu\lambda} - \mathcal{L}_{V_b}\mathcal{L}_{V_a}\, \mathcal{E}_{\mu\lambda}, \tag{0.10}$$

and using the fact that $[V_a\ ,\ V_b] = f^c{}_{ab} V_c$, we have

The mixed Jacobi identities of the Lie bialgebras $(\mathcal{G}\ ,\ \tilde{\mathcal{G}})$

$$f^k{}_{ij}\,\tilde{f}^{ml}{}_k = f^m{}_{ik}\,\tilde{f}^{kl}{}_j + f^l{}_{ik}\,\tilde{f}^{mk}{}_j + f^m{}_{kj}\,\tilde{f}^{kl}{}_i + f^l{}_{kj}\,\tilde{f}^{mk}{}_i, \qquad (0.11)$$

In the same way, one can consider the dual σ-model with background matrix $\tilde{\mathcal{E}}_{\mu\lambda}$ where the group $\tilde{G}$ acts freely on $\tilde{\mathcal{M}}$ and the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are exchanged.

Definition: Lie bialgebra

Definition (Dual Lie algebra)

Let $\mathcal{G}$ be a finite dimensional Lie algebra, and consider its dual $\tilde{\mathcal{G}}$. By definition, an element $\tilde{x} \in \tilde{\mathcal{G}}$ is a linear functional on $\mathcal{G}$, i.e., $\tilde{x}(y) =< \tilde{x}, y >$ for all $y \in \mathcal{G}$.

Definition (Lie bialgebra)

A *Lie bialgebra* $(\mathcal{G}, \tilde{\mathcal{G}})$ is a Lie algebra $\mathcal{G}$ endowed with a linear map $\delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ such that:

(1) the map δ is a one-cocycle on $\mathcal{G}$ with values in $\mathcal{G} \otimes \mathcal{G}$, i.e.,

$$\delta([x, y]) \;=\; [x \otimes 1 + 1 \otimes x \;,\; \delta(y)] - [y \otimes 1 + 1 \otimes y \;,\; \delta(x)], \qquad (0.12)$$

(2)the dual map $[.,.] : \tilde{\mathcal{G}} \otimes \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ defines a Lie bracket on $\tilde{\mathcal{G}}$, i.e.,

$$< [\tilde{x}, \tilde{y}] \;,\; x > = < \tilde{x} \otimes \tilde{y} \;,\; \delta(x) >, \qquad (0.13)$$

Definition: Drinfeld double

Definition (Drinfeld double)

A Drinfeld double is a Lie algebra $\mathcal{D}$ which decomposes into the direct sum, as vector spaces, of two maximally isotropic Lie sub-algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ ($\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$), each corresponding to a Poisson-Lie group (G and $\tilde{G}$), such that the sub-algebras are duals of each other in the usual sense.

Notice that both $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are isotropic with respect to $< \ . \ , \ . \ >$, i.e.,

$$\begin{aligned} < T_a \ , \ T_b > &= < \tilde{T}^a \ , \ \tilde{T}^b > = 0, \\ < T_a \ , \ \tilde{T}^b > &= < \tilde{T}^b \ , \ T_a > = \delta_a{}^b. \end{aligned} \tag{0.14}$$

Lie algebra of Drinfeld double:

$$\begin{aligned} [T_a \ , \ T_b] &= f^c{}_{ab} \ T_c, \qquad [\tilde{T}^a \ , \ \tilde{T}^b] = \tilde{f}^{ab}{}_c \ \tilde{T}^c, \\ [T_a \ , \ \tilde{T}^b] &= \tilde{f}^{bc}{}_a \ T_c + f^b{}_{ca} \ \tilde{T}^c. \end{aligned} \tag{0.15}$$

Consider a two-dimensional sigma model for the d field variables $X^A = (x^\mu, y^i)$ where x^μ $(\mu = 1, ..., dimG)$ are coordinates of Lie group G that act freely from right on manifolds $\mathcal{M} \approx O \times G$. The y^i $(i = 1, ..., d - dimG)$ are coordinates of the orbit O. The corresponding action has the form:

The original sigma model on manifold $\mathcal{M} \approx O \times G$

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^- \Big[E_{ab}(g, y^i)\ R_+^a\ R_-^b + \phi^{(1)}_{aj}(g, y^i) R_+^a \partial_- y^j + \phi^{(2)}_{ib}(g, y^i)\partial_+ y^i R_-^b + \phi_{ij}(g, y^i)\partial_+ y^i \partial_- y^j\Big], \qquad (0.16)$$

where $R^a_\pm$ are the components of the right-invariant Maurer-Cartan one-forms on G:

$$R_\pm = R^a_\pm\ T_a = (\partial_\pm g\, g^{-1})^a\ T_a = \partial_\pm x^\mu\ R_\mu{}^a\ T_a.,$$

Similarly we introduce another σ-model for the d field variables $\tilde{X}^{M} = (\tilde{x}^{\mu}, y^{i})$, where $\tilde{x}^{\mu},\ \mu = 1, \cdots, dim\ \tilde{G}$ parametrize an element $\tilde{g}$ of a Lie group $\tilde{G}$ whose dimension is, however, equal to that of G, and the rest of the variables are the same y^{i}'s used in the original sigma model.

The dual sigma model on manifold $\tilde{\mathcal{M}} \approx O \times \tilde{G}$

$$\tilde{S} = \frac{1}{2}\int d\sigma^{+} d\sigma^{-}\Big[\tilde{E}^{ab}(\tilde{g}, y^{i})\ \tilde{R}_{+a}\tilde{R}_{-b} + {\tilde{\phi}^{(1)}}{}^{a}{}_{j}(\tilde{g}, y^{i})\ \tilde{R}_{+a}\partial_{-}y^{j}$$
$$+ {\tilde{\phi}_{i}^{(2)}}{}^{b}(\tilde{g}, y^{i})\partial_{+}y^{i}\ \tilde{R}_{-b} + \tilde{\phi}_{ij}(\tilde{g}, y^{i})\partial_{+}y^{i}\partial_{-}y^{j}\Big]. \qquad (0.17)$$

where $\tilde{R}_{\pm a}$ are the components of the right-invariant Maurer-Cartan one-forms on $\tilde{G}$:

$(\partial_{\pm}\tilde{g}\ \tilde{g}^{-1})_{a} = \tilde{R}_{\pm a} = \partial_{\pm}\tilde{x}^{\mu}\tilde{R}_{\mu a}.$

It has been shown that the various couplings in the original sigma model action are restricted to be

The various couplings in the original sigma model

$$E = \left(E_0^{-1} + \Pi\right)^{-1}, \qquad \phi^{(1)} = E\ E_0^{-1}\ F^{(1)},$$
$$\phi^{(2)} = F^{(2)}\ E_0^{-1}\ E, \qquad \phi = F - F^{(2)}\ \Pi\ E\ E_0^{-1}\ F^{(1)}, \tag{0.18}$$

Relation between the background matrices of the model and its dual

$$\tilde{E} = \left(E_0 + \tilde{\Pi}\right)^{-1},$$
$$\tilde{\phi}^{(1)} = \tilde{E}\ F^{(1)},$$
$$\tilde{\phi}^{(2)} = -F^{(2)}\ \tilde{E},$$
$$\tilde{\phi} = F - F^{(2)}\ \tilde{E}\ F^{(1)}. \tag{0.19}$$

The Poisson structure on G: $\Pi^{ab}(g) = b^{ac}(g)\ (a^{-1})_c{}^b(g)$

$$g^{-1}\,T_a\ g = a_a{}^b(g)\ T_b, \qquad g^{-1}\,\tilde{T}^a g = b^{ab}(g)\ T_b + (a^{-1})_b{}^a(g)\ \tilde{T}^b. \qquad (0.20)$$

In the non-Abelian T-duality case, where $f^a{}_{bc} \neq 0$ and $\tilde{f}^{ab}{}_c = 0$, it follows that $b(g) = 0$, then, $\Pi(g) = 0$; consequently, $E = E_0$, and thus the original sigma model reduces to

The original sigma model: the non-Abelian T-duality case

$$S = \frac{1}{2}\int d\sigma^+ d\sigma^- \Big[E_{0\,ab}\ R_+^a\ R_-^b + F^{(1)}_{aj} R_+^a \partial_- y^j + F^{(2)}_{ib} \partial_+ y^i R_-^b + F_{ij} \partial_+ y^i \partial_- y^j\Big].$$

$SO(2,2)$ Lie algebra

The general form of $so(2,d)$ Lie algebra

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} - \eta_{ac}M_{bd}. \qquad (0.21)$$

$\eta_{ab} = diag(-1,1,-1), \;\; a,b = 0,1,2:$

$$D = -M_{12}, \;\; K = M_{01} + M_{02}, \;\; P = -(M_{01} - M_{02})$$

$so(2,1)$ Lie algebra

$$[D,K] = K, \quad [D,P] = -P, \quad [P,K] = 2D. \qquad (0.22)$$

$so(2,2)$ Lie algebra ($so(2,2) \cong so(2,1) \oplus so(2,1)$)

$$[D,K] = K, \quad [D,P] = -P, \quad [P,K] = 2D.$$
$$[\bar{D},\bar{K}] = \bar{K}, \quad [\bar{D},\bar{P}] = -\bar{P}, \quad [\bar{P},\bar{K}] = 2\bar{D}. \qquad (0.23)$$

WZW model over $SO(2,2)$ Lie group

we define a non-degenerate invariant symmetric metric on the Lie algbra $\mathcal{G}$ as $\Omega_{ab} =< T_a, T_b >$ which is needed to define the WZW model.

ad-invariant non-degenerate metric over $so(2,2)$ Lie algebra

$$< D, D >=< \bar{D}, \bar{D} >= 1/2, \quad < K, P >=< \bar{K}, \bar{P} >= -1. \tag{0.24}$$

WZW action over G Lie group

$$S_{WZW}(g) = \frac{k}{4\pi}\int_{\Sigma} d\sigma^{+} d\sigma^{-} \Omega_{ab} L_{+}^{a} L_{-}^{b} + \frac{k}{24\pi}\int_{B} d^3\sigma \epsilon^{\gamma\alpha\beta} \Omega_{ad} f_{bc}{}^{d} L_{\gamma}^{a} L_{\alpha}^{b} L_{\beta}^{c}, . \tag{0.25}$$

where $L_{\pm}^{a}$ are the components of the left invariant Maurer-Cartan one-forms on G:
$L_{\pm} = L_{\pm}^{a}\ T_a = (g^{-1}\ \partial_{\pm}\ g)^{a}\ T_a.$

WZW model over $SO(2,2)$ Lie group

The group element

$$g = e^{\gamma_+ K} e^{\phi D} e^{\gamma_- P} e^{\gamma_+ \bar{K}} e^{\phi \bar{D}} e^{\gamma_- \bar{P}} \tag{0.26}$$

The left invariant one-forms

$$\begin{aligned}
L^D_\pm &= \partial_\pm \phi - 2\gamma_- e^{-\phi} \partial_\pm \gamma_+, \qquad & L^{\bar{D}}_\pm &= \partial_\pm \bar{\phi} - 2\bar{\gamma}_- e^{-\bar{\phi}} \partial_\pm \bar{\gamma}_+, \\
L^K_\pm &= e^{-\phi} \partial_\pm \gamma_+, & L^{\bar{K}}_\pm &= e^{-\bar{\phi}} \partial_\pm \bar{\gamma}_+, \\
L^P_\pm &= -\gamma_- \partial_\pm \phi + \gamma_-^2 e^{-\phi} \partial_\pm \gamma_+ + \partial_\pm \gamma_-, \\
L^{\bar{P}}_\pm &= -\bar{\gamma}_- \partial_\pm \bar{\phi} + \bar{\gamma}_-^2 e^{-\bar{\phi}} \partial_\pm \bar{\gamma}_+ + \partial_\pm \bar{\gamma}_-.
\end{aligned} \tag{0.27}$$

The background of $SO(2,2)$ WZW model

$$ds^2_{WZW} = 1/2 \; d\phi^2 - 2e^{-\phi} d\gamma_+ d\gamma_- + 1/2 \; d\bar{\phi}^2 - 2e^{-\bar{\phi}} d\bar{\gamma}_+ d\bar{\gamma}_- \tag{0.28}$$

$$B_{WZW} = e^{-\phi} d\gamma_+ \wedge d\gamma_- + e^{-\bar{\phi}} d\bar{\gamma}_+ \wedge d\bar{\gamma}_- \tag{0.29}$$

The dual model is constructed on a 5-dimensional manifold $\tilde{\mathcal{M}} = O \times \tilde{G}$ with 4-dimensional Abelian Lie group $\tilde{G} = 4A_1$ acting freely on it.

8-dimensional Lie algebra from $\mathcal{D} = (\mathcal{A}_2 \oplus \mathcal{A}_2, 4\mathcal{A}_1)$

$$[T_1, T_2] = T_2, \quad [T_3, T_4] = T_4, \quad [T_1, \tilde{T}^{\,2}] = -\tilde{T}^{\,2},$$

$$[T_2, \tilde{T}^{\,2}] = \tilde{T}^{\,1}, \;\; [T_3, \tilde{T}^{\,4}] = -\tilde{T}^{\,4}, \;\; [T_4, \tilde{T}^{\,4}] = -\tilde{T}^{\,3} \tag{0.30}$$

The original model

The group element

$$g = e^{x_1\, T_1}\, e^{x_2\, T_2}\, e^{x_3\, T_3}\, e^{x_4\, T_4}, \qquad (0.31)$$

The right invariant one-forms

$$R^1_{\pm} = \partial_{\pm} x_1, \qquad R^2_{\pm} = e^{x_1}\, \partial_{\pm} x_2,$$

$$R^3_{\pm} = \partial_{\pm} x_3, \qquad R^4_{\pm} = e^{x_3}\, \partial_{\pm} x_4, \qquad (0.32)$$

The original model

To achieve a σ-model with the background $SO(2,2)$ one has to choose the spectator-dependent matrices in the following form:

Spectator field-dependent matrices

$$E_{0\,ab}(e,y^1)=\begin{pmatrix}0 & -e^{-y_1} & 0 & 0\\ -e^{-y_1} & 0 & 0 & 0\\ 0 & 0 & 0 & -e^{-y_2}\\ 0 & 0 & -e^{-y_2} & 0\end{pmatrix},\; F^{(1)}_{a\,i}=\begin{pmatrix}0 & 0\\ -e^{-y_1} & o\\ 0 & 0\\ 0 & -e^{-y_2}\end{pmatrix}$$

$$F^{(2)}_{i\,a}=\begin{pmatrix}0 & -e^{-y_1} & 0 & 0\\ 0 & 0 & -e^{-y_1} & 0\end{pmatrix},\qquad F_{ij}=\begin{pmatrix}k & 0\\ 0 & \bar{k}\end{pmatrix}. \tag{0.33}$$

Since the dual Lie group has assumed to be non-Abelian, it follows from 0.30 and 0.44 that $\Pi(g)=0$.

The original model

The background of original model

$$ds^2 = k\, dy_1^2 - 2e^{x_1 - y_1}\, dx_1\, dx_2 + \bar{k}\, dy_2^2 - 2e^{x_3 - y_2}\, dx_3\, dx_4 \tag{0.34}$$

$$B = e^{x_1 - y_1} \wedge dx_1\, dx_2 + e^{x_3 - y_2} \wedge dx_3\, dx_4 \tag{0.35}$$

If we introduce the coordinate transformation

coordinate transformations

$$y_1 = \phi,\ e^{x_1} = \gamma_+,\ x_2 = \gamma_-,\ y_2 = \bar{\phi},\ e^{x_3} = \bar{\gamma}_+,\ x_4 = \bar{\gamma}_-, \tag{0.36}$$

then, the dual background turns into $SO(2,2)$ WZW-model.

The dual model

The background of dual model

$$d\tilde{s}^2 = \frac{1}{2}\, dy_1^2 + \frac{2e^{-y_1}}{\Delta}[d\tilde{x}_1\, d\tilde{x}_2 - 2e^{-y_1}\, d\tilde{x}_1\, dy_1] + \frac{1}{2}\, dy_2^2 + \frac{2e^{-y_2}}{\bar{\Delta}}[d\tilde{x}_3\, d\tilde{x}_4 - 2e^{-y_2}\, d\tilde{x}_3\, dy_2], \quad (0.37)$$

$$B = \frac{\tilde{x}_1}{\Delta} d\tilde{x}_1 \wedge [d\tilde{x}_2 - e^{-y_1}\, dy_1] + \frac{\tilde{x}_4}{\bar{\Delta}} d\tilde{x}_3 \wedge [d\tilde{x}_4 - e^{-y_2}\, dy_2] \quad (0.38)$$

Where $\Delta = \tilde{x}_2^2 - e^{-2y_1}$ and $\bar{\Delta} = \tilde{x}_4^2 - e^{-2y_2}$

Dual target space as a 6-dimensional black string

In order to interpret dual geometry, we use the following coordinate transformatins

coordinate transformatins for dual space

$$\tilde{x}_1 = -U - (W + 1/2\ e^{w}),\ \tilde{x}_2 = -e^{T}(1 + e^{-w}),\ y_1 = W - T,$$

$$\tilde{\bar{x}}_1 = -\bar{U} - (\bar{W} + 1/2\ e^{\bar{w}}),\ \tilde{\bar{x}}_2 = -e^{\bar{T}}(1 + e^{-\bar{w}}),\ \bar{y}_1 = \bar{W} - \bar{T}, \qquad (0.39)$$

Also if we use following transformatins:

$$e^{w} = \frac{2}{r-1},\ T = \sqrt{2}(t + \frac{x}{\sqrt{3}}),\ U = (t - \frac{x}{\sqrt{3}})$$

$$e^{\bar{w}} = \frac{2}{\bar{r}-1},\ \bar{T} = \sqrt{2}(\bar{t} + \frac{\bar{x}}{\sqrt{3}}),\ \bar{U} = (\bar{t} - \frac{\bar{x}}{\sqrt{3}}) \qquad (0.40)$$

Dual target space as a 6-dimensional black string

metric

$$d\tilde{s}^2 = \frac{dr^2}{2(r-1)^2} - (1-\frac{2}{r})dt^2 + (1-\frac{2}{3r})dx^2 + \frac{2}{\sqrt{3}}dtdx$$
$$+\frac{d\tilde{r}^2}{2(\tilde{r}-1)^2} - (1-\frac{2}{\tilde{r}})d\tilde{t}^2 + (1-\frac{2}{3\tilde{r}})d\tilde{x}^2 + \frac{2}{\sqrt{3}}d\tilde{t}d\tilde{x} \qquad (0.41)$$

anti symmetric B-field

$$\tilde{B} = -\frac{2}{\sqrt{3}}(2+\frac{1}{r})dt \wedge dx + \frac{2}{\sqrt{3}}(2+\frac{1}{\bar{r}})d\bar{t} \wedge d\bar{x} \qquad (0.42)$$

The metric has singularity at four point; $r=0,\ r=1,\ \bar{r}=0,\ \bar{r}=1$

Dual target space as a 6-dimensional black string

Ricci scalar

$$\mathcal{R} = \frac{4\mathrm{r} - 7}{\mathrm{r}^2} + \frac{4\bar{\mathrm{r}} - 7}{\bar{\mathrm{r}}^2}. \tag{0.43}$$

In addition, it can be shown that the background of the dual model can be an answer to the standard Bosonic supergravity equations.

Torsion

$$\tilde{H} = \frac{2}{\sqrt{3}}[\frac{1}{r^2}\, dt \wedge dr \wedge dx + \frac{1}{\bar{r}^2}\, d\bar{t} \wedge d\bar{r} \wedge d\bar{x}]. \tag{0.44}$$

Finally, we showed that the Beta-function equations are met up to the one-loop with the cosmological constant $\Lambda = 2$, provided that the Dilaton constant is taken as $\Phi = c_0 - \log(r\bar{r})$.

Thank you for your attention